How To Prepare Students for Algebra

By H. Wu

Recent interest in mathematics education has put the teaching of algebra in the national spotlight. The present national goal is not only "Algebra For All," but also "Algebra in the Eighth Grade." Because algebra has come to be regarded as a gatekeeper course—those who successfully pass through will keep going while those who don't will be permanently left behind—the high failure rate in algebra, especially among minority students, has rightfully become an issue of general social concern. Many solutions of a pedagogical nature have been proposed, including the teaching of "algebraic thinking" starting in kindergarten or first grade. I will argue in this paper that no matter how much "algebraic thinking" is introduced in the early grades and no matter how worthwhile such exercises might be, the failure rate in algebra will continue to be high unless we radically revamp the teaching of fractions and decimals.

The proper study of fractions provides a ramp that leads students gently from arithmetic up to algebra. But when the approach to fractions is defective, that ramp collapses, and students are required to scale the wall of algebra not at a gentle slope but at a ninety degree angle. Not surprisingly, many can't. To understand why fractions hold the potential for being the best kind of "pre-algebra," we must first consider the nature of algebra and what makes it different from whole number arithmetic.

Algebra is generalized arithmetic. It is a more abstract and more general version of the arithmetic operations with whole numbers, fractions, and decimals. *Generality* means algebra goes beyond the computation of concrete numbers and focuses instead on properties that are common to all the numbers under discussion, be it positive fractions, whole numbers, etc. In whole number arithmetic, $5 + 4 = 9$, for example, means just that, nothing more, nothing less. But algebra goes beyond the specific case to statements or equations that are true for *all* numbers at *all* times. *Abstraction*, the other characteristic of algebra, goes hand-in-hand with generality. One cannot define abstraction any more than one can define poetry, but very roughly, it is the quality that focuses at each instant on a particular property to the exclusion of others. In algebra, generality and abstraction are expressed in symbolic notation. Just as there is no poetry without language, there is no generality or abstraction without symbolic notation. Fluency with symbolic manipulation is therefore an integral part of proficiency in algebra. I will give an illustration of the concepts of generality and abstraction and how they are served by the use of symbolic notation by considering the problem of when the area of a rectangle with a fixed perimeter is largest. This of course would not be appropriate as an entry-level algebra problem, but we choose it because it is an interesting phenomenon and because it illustrates the nature of algebra well. As preparation, let us begin with some well-known algebraic identities.

If x, y and z are *any* three numbers, then

$$xy = yx$$

and

$$x(y + z) = xy + xz$$

These are called, respectively, the *commutative law* (of multiplication)—which simply means that changing the order of

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factors in multiplication does not change the answer—and the *distributive law,* which means that if a number is broken into parts and each part is multiplied, the answer is the same as when the number is kept whole and multiplied. These laws are usually taken as basic assumptions about *all* numbers, as we do likewise here. Using these simple but powerful mathematical laws, we can derive three general statements about numbers. First, for any numbers a and b, the following identity is always valid

$$(a + b)^2 = a^2 + 2ab + b^2. \tag{1}$$

This is easily derived from the preceding two identities by substituting $(a + b)$ for x , a for y, and b for z:

$$\begin{aligned}
(a + b)^2 &= (a + b)(a + b) \\
&= (a + b)a + (a + b)b && \text{(the distributive law)} \\
&= a(a + b) + b(a + b) && \text{(the commutative law)} \\
&= aa + ab + ba + bb && \text{(the distributive law twice)} \\
&= a^2 + 2ab + b^2 && \text{(the commutative law)}
\end{aligned}$$

Now because identity (1) is valid for *any* two numbers a and b—including negative numbers—we may replace b by its opposite $-b$. When we do, identity (1) becomes:

$$(a + (-b))^2 = a^2 + 2a(-b) + (-b)^2.$$

But $a + (-b) = a - b$, $2a(-b) = -2ab$, and $(-b)^2 = b^2$. Therefore we may rewrite the preceding as:

$$(a - b)^2 = a^2 - 2ab + b^2 \tag{2}$$

Identity (2) could have been derived directly as in identity (1). The present derivation, however, serves the purpose of illustrating one kind of abstract reasoning that routinely enters into algebraic arguments: If a statement is true for *all* numbers, then we can reap dividends by specializing the statement to specific numbers. Let us see, for example, how identities (1) and (2) can be used to compute with ease the squares of numbers close to multiples of 10. Using identity (1), we see that $62^2 = (60 + 2)^2 = 60^2 + (2 \times 60 \times 2) + 2^2 = 3{,}600 + 240 + 4 = 3{,}844$. Or, we can solve a seemingly more difficult problem, $(79)^2$, by first converting 79 to $(80 - 1)$. Then, using identity (2): $79^2 = (80 - 1)^2 = 80^2 - (2 \times 80 \times 1) + 1^2 = 6{,}400 - 160 + 1 = 6{,}241$. On a more sophisticated level, (1) can be used to explain the standard algorithm for the extraction of the square root of a whole number,[1] but we have to omit the details for lack of space.

We need one more identity. Again with a and b as arbitrary numbers, and using only the distributive and commutative laws,

$$\begin{aligned}
(a + b)(a - b) &= a(a - b) + b(a - b) && \textit{(distributive law)} \\
&= a^2 - ab + ba - b^2 && \textit{(distributive law)} \\
&= a^2 - ab + ab - b^2 && \textit{(commutative law)} \\
&= a^2 - b^2
\end{aligned}$$

So we have:

$$a^2 - b^2 = (a + b)(a - b) \tag{3}$$

Note that (3) can also be used to provide a simple way to compute the product of some special numbers, e.g., $89 \times 91 = (90 - 1)(90 + 1) = 90^2 - 1^2 = 8{,}100 - 1 = 8{,}099$. Likewise, $117 \times 123 = (120 - 3)(120 + 3) = 120^2 - 3^2 = 14{,}400 - 9 = 14{,}391$.

With very few exceptions in mathematics, one cannot understand the general without first understanding the particular.

Let us now take up the problem of the areas of rectangles in earnest: We want to show that the area of a rectangle with unequal sides is less than the area of a square with the same perimeter. (Recall that the *perimeter* of a rectangle is the sum of the lengths of all four sides.) Right at the outset, notice how this problem is distinctly different from a typical problem in arithmetic: Instead of saying that the rectangles of sides 1.5 and 2.5, and 1.9 and 2.1 have areas less than that of a square with sides equal to 2 (they all have perimeter 8)—a fact that can be easily verified by simple computations—we are claiming that, *no matter what the lengths of the sides of a rectangle may be,* its area must be less than that of a square with the same perimeter. This is an example of generality: not about one rectangle or a few rectangles, but about *all* rectangles with unequal sides.

With very few exceptions in mathematics, one cannot understand the general without first understanding the particular. Let us, therefore, first verify such a claim about areas of rectangles by considering some concrete cases. Take three rectangles with perimeters equal to 8, say with sides of 1.7 and 2.3, 1.8 and 2.2, and 1.9 and 2.1. Let us check that their areas are less than 4, which is the area of the square with side 2 (and therefore perimeter 8):

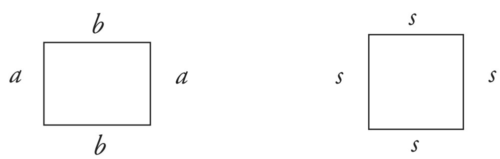
$$1.7 \times 2.3 = 3.91$$
$$1.8 \times 2.2 = 3.96$$
$$1.9 \times 2.1 = 3.99$$

Indeed, they all have areas of less than 4. Psychologically, these numbers give us more confidence in the general case, but mathematically, they would serve no purpose beyond themselves unless we can extract from them a common thread that sheds light on all other rectangles. An inspection of these numbers suggests that as the shorter side of the rectangle increases toward 2 (from 1.7 to 1.8 to 1.9)—so that correspondingly the longer side also decreases toward 2—the area of the rectangle increases toward 4. To put this hypothesis to the test, we look at the areas of rectangles with sides 1.99 and 2.01, and 1.999 and 2.001:

$$1.99 \times 2.01 = 3.9999$$
$$1.999 \times 2.001 = 3.999999$$

The new numerical data, therefore, support this hypothesis. Of course, more numerical data should be compiled if this discussion takes place in a classroom. Certainly rectangles with other perimeters should be considered, for example, those with sides 2.7 and 3.3, 2.8 and 3.2, and 2.9 and 3.1 (all with perimeter 12), etc. They will be seen to give further corroboration of this hypothesis. The numerical evidence therefore suggests that if d denotes the deficit of the shorter side of the rectangle compared with the length of a side of the square with the same perimeter, then as d gets smaller, the area of the rectangle gets bigger. This tells us that we should concentrate on this deficit in developing the general case.

Thus, let the sides of a rectangle be a and b ($a < b$), and let the side of the square with the same perimeter be s:



The area of the rectangle is then ab and the area of the corresponding square is s^2. What we want to show is therefore:

$$ab < s^2. \tag{4}$$

Because the rectangle and the square have the same perimeters, $2a + 2b = 4s$, or what is the same thing, dividing all terms by 2,

$$a + b = 2s. \tag{5}$$

Now define a positive number d as the deficit of a compared with s, i.e., d is the difference between the side of the square and the shorter side of the rectangle:

$$a = s - d$$

Now if a is less than s by the amount d, then the longer side b must exceed s by the same amount because, from (5), a and b must add up to $2s$. Thus,

$$b = s + d.$$

Now that we know that side $a = s - d$, and side $b = s + d$, we can compute the area ab of the rectangle using identity (3), which you will recall, in its general form, is $a^2 - b^2 = (a + b)(a - b)$:

$$ab = (s - d)(s + d) = s^2 - d^2$$

Naturally, $s^2 - d^2 < s^2$ because d^2 is always a positive quantity. Therefore, ab (the area of the rectangle), which we now see is equal to $s^2 - d^2$, must be less than s^2 (the area of the square), and this is exactly what we had set out to prove.

One fact easily stands out in the preceding considerations: Fluency with the basic skills, both at the arithmetic and symbolic levels, is a *sine qua non* of this demonstration. Fluent computation with numbers lies at the foundation of the symbolic manipulations and the ultimate solution because the numerical experimentations furnished the platform to launch the idea of writing ab as $(s - d)(s + d)$. In addition, of course, identity (3), the difference of squares, had to have been at one's fingertips before such an idea would surface in the first place. Now, it could be argued that fluency with arithmetic operations is irrelevant in this discussion because all the numerical evidence we accumulated above could have been easily obtained by use of a calculator. Such an argument may seem to be valid, but it overlooks a hidden factor. If students are not sufficiently fluent with the basic skills to take the numerical computations for granted, either because they lack practice or rely too frequently on technology, then their mental disposition toward computations of any kind would soon be one of apprehension and ultimately instinctive evasion. How, then, can they acquire the necessary confidence to confront the kind of symbolic computations associated with identities (1)-(3)? In other words, is it reasonable to expect a person to run well if his walk is wobbly?

Having made the point that computational facility on the numerical level is a prerequisite for facility on the symbolic level, we must not oversimplify a complex issue by equating the two kinds of facility. There is a sizable distance between them, and students in arithmetic need a gradual acclimatization with the concepts of generality and abstraction before they can learn to compute on a symbolic level. In terms of the school curriculum, we can describe this progression in greater detail. It is difficult to teach students in whole number arithmetic about symbolic notation other than to write down in symbolic form the commutative laws, the distributive law, etc., because the basic computational algorithms for whole numbers do not lend themselves to be explained symbolically. However, the subject of fraction arithmetic—usually addressed in grades 5 and 6—is rife with opportunities for getting students comfortable with the abstraction and generality expressed through symbolic notation. Consider for example the addition of fractions. If one stays away from the concept of the lowest common denominator—a topic we will discuss later on—then for whole numbers a, b, c, and d, the following is true:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \tag{6}$$

Pedagogically, we can approach this formula in the following way: Once the concept of addition for fractions has been clearly defined, then this formula could be shown to be true, first for small numbers such as $a = 1$, $b = 3$, $c = 2$, and $d = 5$, and then for larger numbers such as $a = 3$, $b = 12$, $c = 5$, and $d = 18$. After a sufficient amount of practice, the proof of (6) for *arbitrary* whole numbers $a, b, c,$ and d can eventually be given. Here is an abstract situation where students can slowly build up their intuition from concrete cases to the general case, thereby gaining a gentle introduction to symbolic computations. The importance of good teaching in fractions as an introduction to algebra does not stop here, however. As students get to understand the division of fractions so that $\frac{a}{b}$ becomes meaningful even when a and b are now themselves fractions, one goes on to prove that formula (6) remains valid, *as it stands,* when $a, b, c,$ and d are fractions. Then it follows that (6) is also valid for finite decimals. One can go further. A standard topic in algebra is *rational expressions,* which are quotients of the form $\frac{a}{b}$, where a and b are now polynomials in a variable x, such as $a = x^3 - 3x + 4$ and $b = 5x^2 + 2$. Then the addition of rational expressions is also given by (6) for *polynomials, $a, b, c,$ and d.*

The message is now clear: Formal abstraction is at the heart of algebra. The addition-of-fractions formula (6) is an example because the same formula is seen to encode seemingly disparate information. If we look at the school mathematics curriculum longitudinally, the development of formula (6) from whole numbers a, b, c, d to polynomials as described above takes place over a period of three to four years, and it starts with the teaching of fractions. Without this foundation in fractions, students who come to the study of rational expressions in algebra are severely handicapped.

The importance of fluency with symbolic computations in algebra can be reinforced from a slightly different angle. Let us revisit the problem about the area of rectangles and give it a new proof. Still assuming that the sides of the given rectangle are a and b, with $a < b$, we wish to show as before that its area is less than the area of the square with the same perimeter $2a + 2b$. Again let s be a side of the square in question. Then, because the rectangle and square have the same perimeter, we have $2a + 2b = 4s$, so that, dividing all terms by 4, $s = \frac{1}{2}(a + b)$. Now the area of the rectangle is ab and that of the square is $s^2 = [\frac{1}{2}(a + b)]^2 = \frac{1}{4}(a + b)^2$. What we want to prove, in symbolic language, is that $ab < \frac{1}{4}(a + b)^2$, which, multiplying by 4, can be rewritten as:

$$4ab < (a + b)^2 \qquad (7)$$

Why should we believe (7) is true for any two numbers a and b? Let us try some special cases. If $a = 2$ and $b = 3$, (7) says $24 < 25$; if $a = 5$ and $b = 2$, (7) says $40 < 49$; if $a = 4$ and $b = 11$, (7) says $176 < 225$; and if $a = 7$ and $b = 9$, then (7) says $252 < 256$. These are all true, of course. Let us also try some small numbers: if $a = \frac{1}{2}$ and $b = 3$, (7) says $6 < (3\frac{1}{2})^2$, which is true because $6 < 9 = 3^2 < (3\frac{1}{2})^2$. If $a = \frac{1}{3}$ and $b = \frac{1}{10}$, then (7) says $\frac{2}{15} < (\frac{13}{30})^2$.

This is a more difficult case. There is no way, even for an experienced mathematician, to take a quick glance at $\frac{2}{15}$ and $(\frac{13}{30})^2$ and determine which is larger. Furthermore, it does us no good to compute the square of $\frac{13}{30}$ because then we wind up with a fraction even less receptive to intuition. So what we do is look for something a little easier to work with. For example, instead of $(\frac{13}{30})^2$, we will try $(\frac{12}{30})^2$. If $(\frac{12}{30})^2$ proves to be larger than $\frac{2}{15}$, then so of course will $(\frac{13}{30})^2$. Now, $(\frac{12}{30})^2$ can quickly be simplified to $(\frac{2}{5})^2$, which computes to $\frac{4}{25}$. Then by converting $\frac{2}{15}$ to $\frac{4}{30}$ to make the comparison more obvious, we can easily see that $\frac{4}{25} > \frac{4}{30}$ and therefore $\frac{4}{25} > \frac{2}{15}$. So, we have now shown that (7) is true for this more difficult case. (Of course one could have checked $\frac{2}{15} < (\frac{13}{30})^2$ directly by pushing buttons on a calculator, converting the fractions to decimals and then comparing them, but if you want to see in a substantive mathematical context what estimation can do for you, this is a good example.) So we have some evidence that (7) must be true, although it may be difficult to see from these computations *why* it is true. We recall, however, that identity (1) gives a different expression to the right side of (7). Why not make use of (1) and see if we can simplify (7) to the point where we know what to do next. By (1), the right side of (7), which is $(a + b)^2$, computes to $a^2 + 2ab + b^2$. Thus (7) is the same as $4ab < a^2 + 2ab + b^2$. This is, of course, the same as

$$a^2 + 2ab + b^2 > 4ab \qquad (8)$$

So proving that (7) is true is the same as proving that (8) is true. If we subtract $4ab$ from both sides of (8), then we would arrive at

$$a^2 - 2ab + b^2 > 0 \qquad (9)$$

Conversely, if (9) is true, then (8) would also be true, because by adding $4ab$ to both sides of (9) we would obtain (8). It follows that our task of proving the truth of (7) has been reduced to proving the truth of (9). If we now recall identity (2), which is $(a - b)^2 = a^2 - 2ab + b^2$, then (9) is obviously true because

$$a^2 - 2ab + b^2 = (a - b)^2 > 0.$$

This is so because, as readers will recall, b is greater than a, and so $a - b$ yields a negative number. And since the square of any negative number always produces a positive number, $(a - b)^2$ has to be greater than zero. So we have again proved the desired assertion that the area of a rectangle with unequal sides is less than the area of a square with the same perimeter.

This proof differs from the previous one in its greater reliance on formal symbolic manipulations. If you retrace the steps from (7) to (8) to (9), and then onto the final step of invoking identity (2), you would recognize that those computations are quite different from the ordinary numerical computations one encounters in both whole number and fraction arithmetic. The former depends more on detecting formal patterns of the kind exemplied by identities (1)-(3) and less on brute-force calculations. *This kind of skill is important in algebra,* and students need lots of practice with simpler symbolic computations before they can deal with problems of this nature. It is in this context that we can put in perspective the recent attempt at solving the algebra-learning problem by introducing "algebraic thinking" in the early grades. At the risk of slight oversimplification, we can say that the main characteristic of the "early algebraic thinking" approach is to focus exclusively on the conceptual-understanding aspect of abstract symbols. Such an approach re-

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lies heavily on the use of concrete objects, "real-life examples," and graphs as aids to help students come to grips with the abstract reasoning in algebra. This is an important first step in the learning of algebra, but the learning must go on to encompass the skill component as well, i.e., the mastery of symbolic computation. Anecdotal evidence abounds of students who can demonstrate a conceptual understanding of the use of symbols but who nevertheless fail to manipulate them correctly in computations. Is there, perhaps, a danger that the "early algebraic thinking" approach would be taken by teachers (and therefore by students as well) as the *only* step needed to prepare students for algebra? In the absence of firm data, one can only offer an educated guess: Such a danger is very real because, in the words of Roger Howe of Yale University, "You have elementary school teachers who do not know what algebra is about, so they're not in the position to think about how the arithmetic they're teaching will mesh with algebra later."[2]

How would the good teaching of fractions help students acquire the symbolic computational skills necessary for success in algebra? The addition of fractions was presented earlier as an example, but that is a small example. A more substantial example is how the well-known cross-multiply algorithm can be used to advantage for this purpose. Of course, when students are first taught the cross-multiply algorithm, they would use concrete numbers not abstract symbols. But at some point—perhaps sixth grade—they need to be introduced to the symbolic representation of this algorithm and its proof. This, again, prepares them for the concept of generality in algebra. The cross-multiply algorithm asserts that the equality of two fractions

$$\frac{a}{b}=\frac{c}{d}$$

(where a, b, c, d are whole numbers) is the same as the equality of a pair of whole numbers

$$ad = bc.$$

The reason is very simple: By the equivalence of fractions, we have

$$\frac{a}{b}=\frac{ad}{bd} \text{ and } \frac{c}{d}=\frac{bc}{bd}.$$

Therefore, the equality

$$\frac{a}{b}=\frac{c}{d}$$

is the same as

$$\frac{ad}{bd}=\frac{bc}{bd},$$

which is therefore the same as $ad = bc$.

Note first of all that the preceding proof uses symbolic notation. The other thing of note is that this algorithm seems to get caught in the crossfire between two schools of thought. On the one hand, the older curricula tend to ram the algorithm down students' throats with little or no explanation but otherwise make use of it quite effectively. On the other hand, the more recent curricula would try to make believe that there is no such algorithm, and would at best hold it gingerly at arm's length. Both are defective presentations of a piece of useful mathematics. Let us illustrate a good application of this algorithm by proving:

$$\frac{a}{b}=\frac{c}{d} \text{ is the same as } \frac{a}{a+b}=\frac{c}{c+d} \tag{10}$$

You may be wondering why you should be interested in such an arcane statement. Because there is no point in explaining something you don't care about, let me begin by showing you its usefulness.

Consider a standard problem: If the ratio of boys to girls in an assembly of 224 students is 3:4, how many are boys and how many are girls? This is an easy problem, but what is important is that we are going to present a solution to this problem using (10), which is strictly mathematical and free of any psychological overtone connected with the concept of a "ratio." Here "ratio" would mean *division*, and just that. No more and no less. So the given data that the ratio of boys to girls being 3:4 means exactly that if B denotes the number of boys and G denotes the number of girls in the audience, then

$$\frac{B}{G}=\frac{3}{4}$$

(We have just made use of the *provable* interpretation of a fraction $\frac{a}{b}$ as "a divided by b.")

Let us proceed. According to (10), we now also know that $\frac{B}{B+G}=\frac{3}{3+4}$. We are given that $B + G = 224$ and of course $3 + 4 = 7$. So

$$\frac{B}{224}=\frac{3}{7},$$

and from this one readily solves for $B = 96$. There are 96 boys and, therefore, $224 - 96 = 128$ girls.

If you are now convinced that (10) may be interesting, it is time to prove its validity. By the cross-multiply algorithm,

$$\frac{a}{a+b}=\frac{c}{c+d}$$

is the same as $a(c + d) = (a + b)c,$ which upon expanding both sides using the distributive law becomes $ac + ad = ac + bc$. Taking ac away from both sides, we are left with $ad = bc,$ so the equality

$$\frac{a}{a+b}=\frac{c}{c+d}$$

is the same as $ad = bc.$ But the cross-multiply algorithm also tells us that $ad = bc$ is the same as $\frac{a}{b}=\frac{c}{d}$. So (10) is now fully justified.

The proof of a statement such as (10) is the kind of lesson that should be a regular part of the teaching of fractions. It is not only a useful piece of mathematical information, but also—and this is important to our argument here—it exposes students to a small amount of symbolic computation naturally. If fractions are taught properly, how can one hope for a better preparation for algebra? Unfortunately, the state of the teaching of fractions is anything but proper at the moment. If we believe that the subject of mathematics is a logical unfolding of ideas starting with clear and precise definitions and assumptions, then mathematics education in grades five through seven—where the teaching of fractions and decimals dominates—has not been about mathematics for quite some time.

It is impossible to catalogue all the wrongs in the way fractions are taught, in all kinds of curricula, in a few paragraphs. Perhaps we can give two clear cut examples. The first transgression is that a fraction is never defined in textbooks or professional development materials. We have children who are completely lost as to what a fraction is, and educators who publicly bemoan students' failure to grasp the concept. Yet strangely enough, no clear definition of a fraction is ever offered. It is sobering to realize that in elementary education, the importance of having precise definitions of key concepts such as fractions or decimals is not recognized.

The pedagogical problem is, in fact, far worse than this, because it is not only that a fraction is never defined but that very confusing information is impressed on the children. First, children are told that a fraction such as ⅗ is an activity: When they see a pie, if they slice it into 5 equal parts and take 3 of them, what they get will be ⅗ of the pie. They can do the same to an apple, a square, etc. The problem is that if a fraction is an activity, how to tell a child to add or divide two activities? Second, children are told that a fraction is a very complicated concept and they must know that the symbol ⅗ comes equipped with many interpretations. It is 3 parts of a division into 5 equal parts; it is 3 "divided" by 5 (students understand that 10 divided by 5 is 2, but 3 divided by 5?); it is an operation that reduces the size of anything from 5 to 3, and it is also a "ratio" of 3 to 5. At this point, it is fair to say that learning fractions ceases to be a mathematical exercise because what is required is not intellect but an uncommon supply of faith.

With the proper infusion of precise definitions, clear explanations, and symbolic computations, the teaching of fractions can eventually hope to contribute to mathematics learning in general and the learning of algebra in particular.

A second example is the addition of fractions. What would a child experience when she is exposed to a typical lesson on adding fractions? Because she already knows how to add whole numbers—where intuition is strongly grounded on the counting on her fingers—she expects the addition of fractions to be similar. But then she is told that adding ¾ to ⅙ requires finding the least common multiple of 4 and 6, which is 12. Then she is supposed to change ¾ to 9⁄12 and ⅙ to 2⁄12, and add 9⁄12 to 2⁄12 by adding only the numerators, thereby obtaining.

$$\frac{3}{4}+\frac{1}{6}=\frac{9}{12}+\frac{2}{12}=\frac{11}{12}.$$

This kind of education completely disrupts a child's normal mathematical development. Instead of building on what she knows about the addition of whole numbers—as it should—this "explanation" confuses her by instilling the false belief that whole numbers and fractions are completely different objects.

Some of the more recent curricula have improved on this dismal situation by making better sense of adding fractions. Where they still fail is in not having formulated a clear definition of a fraction that includes whole numbers as a special kind of fraction. As a result, students do not see that there is a smooth continuum from whole numbers to fractions. A more serious concern is the failure of the newer curricula to emphasize the computational algorithms such as formula (6) for the addition of any two fractions. In these curricula, adding fractions remains a "conceptual" preoccupation: Understanding the *idea,* the *concept,* is deemed sufficient. Being able to fluently execute the operations until they become second nature and thus effortlessly available when needed is downplayed. We have said it once before, but we should say it again: Fluency in computation is very important for the learning of algebra, and formulas such as (6) provide conceptual continuity between grades. If we are allowed to look further ahead, we can say that the computational aspect of numbers is essential for the learning of both higher mathematics and science.

Grades five through seven are supposed to prepare students for algebra. But children who come through two or three years of the usual kind of instruction in fractions are in reality refugees from an educational devastation. Mathematically starved and intellectually demoralized, they harbor a deep distrust of mathematics as a whole. How, then, do we expect them to learn algebra?

We have not dealt with decimals thus far, but the problems there are entirely parallel to those in fractions. Students are generally not told, forcefully and clearly, that (finite) decimals are merely a shorthand notation for a special type of fractions, namely, those whose denominators are 10, 100, 1,000, or, more generally, a power of 10. Therefore, 0.12 is nothing but an alternate notation for $\frac{12}{100}$, as 1.76 is for $1 + \frac{76}{100}$. The failure to provide a clear definition of a decimal leaves students groping in the dark for the meaning of this mysterious piece of notation.[3] No wonder they resort to such wild guesses as $0.19 > 0.4$ on account of the fact that $19 > 4$. A clear definition of decimals would also help explain the usual rules about "moving the decimal point," e.g., $0.5 \times 0.43 = 0.215$ because we can see in a straightforward manner that

$$0.5 \times 0.43 = \frac{5}{10} \times \frac{43}{100} = \frac{5 \times 43}{10 \times 100} = \frac{215}{1000} = 0.215.$$

There is no need to memorize this rule by brute force, not here and not anywhere else in mathematics. Incidentally, notice how the understanding of decimals is founded on an understanding of fractions.

The mathematical defects in the usual presentation of fractions and decimals can be remedied in a straightforward manner without appealing to any heroic measures. Details are not called for in an article of this nature, but it would be appropriate to mention briefly that one can, for example, begin with a definition of a *number* (which includes whole numbers and fractions) as a point on the number line. Of course, to do so would require that the number line be introduced early, say in the third and fourth grades. One could then *gradually but carefully* raise the level of abstract reasoning and increase the use of symbolic computations to explain the more subtle aspects of fractions, such as the interpretation of fractions as quotients, as well as the more formal concepts such as the division and multiplication of fractions or operations with complex fractions.[4] With the proper infusion of precise definitions, clear explanations, and symbolic computations, the teaching of fractions can eventually hope to contribute to mathematics learning in general and the learning of algebra in particular.

It remains to supplement these curricular considerations of mathematics in grades five through seven with two observations. One is the glaring omission thus far of the basic reason why fractions are critical for undertanding algebra: The study of linear functions, which is the dominant topic in beginning algebra, requires a good command of fractions.[5] The slope of the graph of a linear function is by definition a fraction, to cite just one example. The solution of simultaneous linear equations leads inevitably to the use of fractions, to cite another. Thus on the skills level alone, there is no escape from fractions in algebra.

The other observation is that no matter what the curricular improvement may be, its implementation rests ultimately with the teacher in the classroom. Liping Ma's pathbreaking book, *Knowing and Teaching Elementary Mathematics,* did away with the myth that elementary mathematics is simple. Nowhere is Ma's observation more apparent than in the teaching of fractions. Fractions are difficult not only for students, but also for their teachers, who, for the most part, are themselves the victims of poor mathematics education.

This, then, brings us full circle. If we are to prepare millions of students to successfully open the gate of algebra, we must prepare their teachers as well. This will require that college math courses for prospective teachers be drastically overhauled, so that they directly address teachers' *mathematical* needs in the classroom. For those already teaching, we will need a massive commitment to inservice development, with classes that do not waste teachers' valuable time. In addition, we should allow teachers who like math to specialize in the field at an earlier grade level. This specialization could begin at grade five—when fractions are introduced—or even earlier, as is done in many other countries. As Richard Askey pointed out in the pages of this magazine two years ago, the use of math specialists would unburden other teachers from a task many of them "now find difficult and unpleasant."[6]

All this can be done. It will require resources, good will, and political resolve. Whether these are forthcoming depends on how seriously we take the slogan "Algebra for All." □

References

[1] Cf. Askey, R.A. (1995) Cube Root Algorithms, *Mathematics in School,* Vol. 24, 42-43. Sad to say, this algorithm has completely disappeared from school mathematics curricula.

[2] Quoted in Hoff, D.J., Introduction to Algebra: It's Elementary, (2001) *Education Week,* March 28.

[3] Many textbooks introduce third or fourth grade students to decimals as "numbers with a decimal point" without explaining what a decimal point is.

[4] See, for example, Wu, H. (2001) Fractions (Draft), http://www.math.berkeley.edu/~wu/.

[5] Unfortunately, students actually need a much more sophisticated version of the theory of fractions in algebra than the one currently being taught in K-12; they must know the meaning of $\frac{a}{b}$, and how to work with it when a and b are allowed to be irrational numbers. This issue has been systematically ignored in the mathematics curriculum of K-12.

[6] Askey, R.A. (1999) Knowing and Teaching Elementary Mathematics, *American Educator,* Vol. 23, No. 3.