

Mathematics for Teaching: Then and Now

Many readers wrote to us requesting the answers to the questions from the State of California's teacher certification exam of December 1874 that appeared in the Fall 2005 issue. American Educator thanks Richard Askey, a professor of mathematics with the University of Wisconsin-Madison for providing the answers and explanations presented below.

—EDITOR

ARITHMETIC

1. Divide 8,786,742 by the factors 7, 5, and 2. Explain the principle of obtaining the true remainder.

Answer: I assume the problem can be done in any order one wants, so we will first divide 8,786,742 by 2. The remainder is 0. When one divides by 5, the result is the same as dividing the original number by 10, which gives 878,674 and $\frac{2}{10}$. When one divides this by 7 the remainder when dividing 878,674 by 7 is 6 so to find the true remainder when dividing the original number by 70 one has to add $\frac{6}{7} + \frac{2}{70}$ and write this as something divided by 70. That something is 62 which is the true remainder. The point of doing this in steps is that one can divide by 7 and keep track of the remainder without as much work as you would have had if you divided by 70. With a little practice one can do the first mentally, and the second takes longer to learn how to do it mentally since one has to keep two digit numbers in mind, not single digits.

2. What is the greatest common divisor and the least common multiple of the numbers 18, 36, and 24? Explain the principle of obtaining each.

Answer: The greatest common divisor of a set of relatively small whole numbers can be found by factoring them and multiplying the common factors. Here the factors are:

$$18 = 2 \times 3 \times 3$$

$$36 = 2 \times 2 \times 3 \times 3$$

$$24 = 2 \times 2 \times 2 \times 3$$

so the greatest common factor is $2 \times 3 = 6$ since 2 appears only once in the prime factorization of 18 and 3 only appears once in the factorization of 24. The least common multiple is $2 \times 2 \times 2 \times 3 \times 3 = 72$ since 2 appears three times in the factorization of 24 and 3 appears twice in the factorization of both 18 and 36.

More was taught about finding greatest common divisors in 1874 than just prime factorization. If one wants to find the greatest common factor of 299 and 351, where the prime factorization is not obvious, one writes $351 = 299 + 52$, and any number that divides 351 and 299 will have to divide 52. Also any number that divides 299 and 52 has to divide 351, so the greatest common divisor (GCD) of 351 and 299 is the same as the GCD of 299 and 52. What was really done was to divide 299 into 351. Now do this with 52 and 299 getting $299 = 5 \times 52 + 39$. By inspection, the GCD of 39 and 52 is 13, so the greatest common divisor of 299 and 351 is 13. One can check that $299 = 13 \times 23$ and

$351 = 13 \times 27$. This argument works no matter how large the numbers are, while prime factorization becomes harder when the numbers get large and impossible in a practical sense when the numbers are very large.

3. What is the difference between 2 miles, 5 furlongs, 6 rods, 3 yards, 2 feet, 7 inches, and 7 furlongs, 39 rods, 4 yards, 2 feet, 8 inches? Prove your work by subtraction in decimals.

Answer: 2 miles, 5 furlongs, 6 rods, 3 yards, 2 feet, 7 inches minus 7 furlongs, 39 rods, 4 yards, 2 feet, 8 inches can be computed the same way one does subtraction in the decimal system. One needs to convert a large unit to a smaller one if the subtraction at any place would lead to a negative number. If one looks at each place, you see that this needs to be done at each place, so let us start from the left and convert a step at a time. We will do the conversions when moving from left to right, so 2 miles is 1 mile and 8 furlongs, then 8 furlongs plus 3 furlongs is 11 furlongs or 10 furlongs plus 40 rods, etc. We will also use 1 rod is 5.5 yards, or 5 yards, 1 foot and 6 inches.

$$\begin{array}{r} 1 \text{ mi. } 12 \text{ fur. } 45 \text{ rods } 8 \text{ yd } 3 \text{ ft } 13 \text{ in} \\ - 7 \text{ fur. } 39 \text{ rods } 4 \text{ yd } 2 \text{ ft } 8 \text{ in} \\ \hline 1 \text{ mi. } 5 \text{ fur. } 6 \text{ rods } 4 \text{ yd } 1 \text{ ft } 5 \text{ in} \end{array}$$

To "prove" the answer using decimals, one converts everything to feet plus inches or to inches and does the subtraction. This will be left to the diligent reader.

4. Write in words and analyze the following fractions: $\frac{3}{4}$, $2\frac{1}{3}$, $.3$, and $.00007$.

Answer: $\frac{3}{4}$ is three-fourths, or $\frac{75}{100}$ which is often written as $.75$. $2\frac{1}{3}$ is two and one-third, or $\frac{7}{3}$ (seven-thirds). $.3$ is $\frac{3}{10}$ or three-tenths. $.00007$ is $\frac{7}{100000}$ or 7 hundred-thousandths.

5. If $\frac{3}{8}$ of 7 tons of coal cost $\$93\frac{1}{3}$, what will $\frac{3}{4}$ of 5 tons cost? Work by analysis and prove by proportion.

Answer: If $\frac{3}{8}$ of 7 tons of coal costs $\$93\frac{1}{3}$, or $\$280\frac{2}{3}$, then $\frac{1}{8}$ of 7 tons costs $\$140\frac{2}{3}$. Seven tons costs $\$140$, so 1 ton costs $\$20$ and 5 tons costs $\$100$. Three-fourths of that is $\$75$.

6. What will it cost to build a wall 650 feet long, 8 feet high, and 2 $\frac{2}{3}$ feet thick, at $\$9.75$ per 1,000 bricks—each brick being 8 inches long, 4 inches wide, and 2 inches thick?

Answer: In this problem we will just figure out the volume of the wall, find out how many bricks are needed to fill this volume and then use the cost of bricks. Of course there should be room left for mortar, added bricks because some

Note: 1 mile = 5,280 feet or 1,760 yards or 8 furlongs; 1 furlong = 660 feet or 220 yards or 40 rods; 1 rod = 16.5 feet or 5.5 yards; 1 acre = 43,560 square feet or 4,840 square yards or 160 square rods.

might break, and the cost of labor, none of which can be determined from the given data. The volume of the wall is $650 \times 8 \times 2\frac{1}{3}$ cubic feet, so is $650 \times 6\frac{2}{3}$ cubic feet. The volume of each brick is $\frac{2}{3} \times \frac{1}{3} \times \frac{1}{6}$ cubic feet, or $\frac{1}{27}$ cubic feet. The total number of bricks needed is then $650 \times 64 \times 9$ when the division is done. Then divide this by 1,000 and multiply by \$9.75 to get the total cost of the bricks. I get \$365.04.

7. A grocer sold 4 barrels of sugar for \$30 each; on two barrels he gained 20 percent, on the other two barrels he lost 20 percent; did he gain or lose on the whole?

Answer: If a grocer sells four barrels of sugar for \$30 each, makes 20% profit on two and loses 20% on the other two, his original costs can be figured by computing x times $1.20 = 30$ for the cost of one of the barrels where he made a profit and x times $.80 = 30$ for one of the barrels which were sold at a loss. The original cost of a barrel sold at a profit is $\frac{30}{1.2} = \frac{300}{12} = \frac{100}{4} = \25 , and the original cost of a barrel sold at a loss is $\frac{30}{.8} = \frac{300}{8} = \37.50 . Thus his gain was \$10 (on the two barrels sold at a profit) and his loss was \$15 so he lost \$5 on the sale. The question was whether he gained or lost and that can be solved without doing the calculations explicitly. His initial cost of a barrel which was sold at a loss was more than \$30 and the initial cost of a barrel which was sold at a profit was less than \$30. 20% of something more than \$30 is more than 20% of something less than \$30 so he lost money.

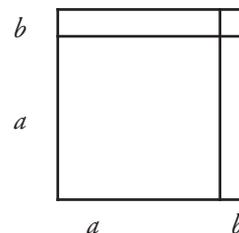
8. A man received \$16.12½ interest on a sum of money, which had been loaned 3½ years at 9 percent. What was the principal?

Answer: Interest in old school textbooks was simple interest over the course of the loan. Thus 9% interest over 3½ years is 9 times $\frac{1}{3}$ or 30%. If the gain was \$16.12½, then the principal was $\frac{16.125}{.3} = \frac{161.25}{3} = 53.75$ so \$53.75.

9. A flagstaff 75 feet high stands in the center of a square lot containing two acres; what is the length of a rope extending from one corner to the top of the staff?

Answer: It is easier to work this problem in yards rather than feet since the numbers are smaller. The area of the square is 2×4840 square yards so a side of the square is the square root of this number. The diagonal of the square is the square root of two times this. The distance from the center of the square, where the flag poll is set, to a corner is half of this and so is the square root of 4840 since square root of 2 times itself is 2. The flag poll is 25 yards high, so when the Pythagorean theorem is used the square of the length of the rope is $4840 + 625 = 5465$. The square of 74 is 5476 so this is a reasonable number to give for the length of the rope. One method of taking square roots by hand was taught in the 1870s, and for many decades. Until about 1915 there were nice explanations why this method worked. When I learned this method in the 1940s it was as a procedure which was illustrated but not explained. The explanation given in most of the arithmetic books from 100 plus years ago came from drawing a square and breaking it up into two smaller squares and two rectangles. Another way of thinking about this method is to recall an important

fact from algebra: $(a+b)^2 = a^2 + 2ab + b^2$ where a^2 means a times a . Rewrite this as $(10a+b)^2 = 100a^2 + (20a+b)b$. The method can be illustrated by looking for the square root of 5465. $70^2 = 4900$ and $80^2 = 6400$ so the square root is between 70 and 80. Subtract $5465 - 4900 = 565$. As a trial divisor divide 565 by $20 \times 7 = 140$. The result is a bit more than 4, but we will have to add 4 to 140 and $4 \times 144 = 576$ which is larger than 565 so the first two digits of the square root of 5465 are 73. If you want a more accurate approximation, continue using 5465.00 or 5465.0000. The trial division of 74 is a reasonable approximation since the actual division is just a little above what we want; 73.9 is the next approximation. The figure which was used to illustrate this method is



but with specific numbers in place of a and b , since this was taught before letters were introduced to stand for general numbers.

10. Extract the cube root of $\frac{9}{21\frac{1}{3}}$

Answer: The cube root of $\frac{9}{21\frac{1}{3}}$ can be found by first sim-

plifying the complex fraction, using $\frac{64}{3}$ for $21\frac{1}{3}$ and multiplying the fraction by $\frac{\frac{3}{64}}{\frac{3}{64}}$ to get $9 \times \frac{3}{64} = \frac{27}{64}$.

$27 = 3^3$ and $64 = 4^3$ so the cube root is $\frac{3}{4}$.

11. What are the names of the units used in the metric system for length, surface, solid, capacity, and weight?

Answer: This is a sad problem since we should have changed to the metric system long ago. In my childhood there was nothing in the arithmetic books about the metric system, but there is now. The hope is that when children who have learned the metric system in school grow up they will be more willing to change. The fact that teachers in 1874 were expected to know the metric system suggests that the current hope might not bear fruit. The names are

length: meter

surface: square meter

solid: cubic meter

capacity: liter

weight: gram

with all of the prefixes to multiply or divide by powers of 10.

MENTAL ARITHMETIC

1. Paid \$2.50 for 5 yards of ribbon, at $12\frac{1}{2}$ cents per

yard, and 3 books at $37\frac{1}{2}$ cents each. How much change did I receive back?

Answer: Combine 1 yard of ribbon and 1 book to get a charge of 50 cents, so \$1.50 is paid for 3 yards of ribbon and 3 books. The remaining ribbon costs 25 cents so you receive 75 cents in change.

2. What percent of 60 is 12?

Answer: 12 is $\frac{1}{5}$ of 60 and $\frac{1}{5}$ is 20%.

3. How many men can perform the same amount of work in 12 days that 6 men can in 4 days?

Answer: There is an unstated assumption that each man does the same amount of work in a given time. With this assumption, you observe that 12 days is three times as long as 4 days so it will take $\frac{1}{3}$ as many men to do the same amount of work. Thus 2 men can do in 12 days what 6 men can do in 4.

4. If a man travels 1 mile in 20 minutes, how many hours and minutes will it take him to travel 17 miles?

Answer: If a man travels 1 mile in 20 minutes, he travels 3 miles in 1 hour, so 15 miles in 5 hours and the remaining 2 miles in 40 minutes. Again there is the unstated assumption that travel is done at the same rate.

5. If A and B can do a piece of work in 4 days, and A can do it alone in 6 days, how long would it take B to do it?

Answer: A can do a job in 6 days so in one day A can do $\frac{1}{6}$ of the job. A and B can do $\frac{1}{4}$ of the job in one day, so B can do $\frac{1}{4} - \frac{1}{6} = \frac{1}{12}$ of the job in one day. Thus it will take B 12 days to do the job. As we have seen, there are some implicit assumptions, constant rate of work for each person whether they are working alone or with someone else.

6. A man being asked how many sheep he had, replied that if he had one and one half times as many more, and $2\frac{1}{2}$ sheep, he would have fifty. How many had he?

Answer: If he had 50 sheep when he had one and a half times as many as he has he would have had 20 sheep, since one and a half times 20 is 30. When the $2\frac{1}{2}$ sheep are considered as he did, he had 19 sheep since one plus one and a half is $2\frac{1}{2}$.

7. A man after spending one half of his money and one third of the remainder, had \$10 left. How much had he at first?

Answer: He had \$15 before spending $\frac{1}{3}$ of the remainder and so started with \$30.

8. How many car tickets, at $6\frac{1}{4}$ cents each, can be purchased for \$7.50?

Answer: If a ticket costs $6\frac{1}{4}$ cents, 4 tickets cost 25 cents, and 25 times 30 is 750 so for \$7.50 it is possible to buy 4 times 30 or 120 tickets.

9. How long will it take a man to clear \$100 if he gets

\$1.50 a day, and pays \$4 a week for board?

Answer: A man who gets \$1.50 a day gets \$9.00 a week if we assume a 6 day work week, which was likely the case in 1874. The \$4 a week for board reduces the weekly take to \$5, so it would take 20 weeks to clear \$100. The calculations are a bit messier if one assumes a 5 day or a 7 day work week.

10. If you get $2\frac{1}{2}$ credits each for 8 questions out of 10, what will be your percentage?

Answer: This problem can only be worked when one has an additional piece of information. In the information about this exam, the total credits for Mental Arithmetic is 50. Then $2\frac{1}{2}$ credits on 8 problems gives 20 credits, so a score of 40%.

ALGEBRA

1. What is a reciprocal? Zero power? Negative exponent?

Answer: Reciprocal: If a is not zero, then its reciprocal, denoted by a^{-1} is the number which satisfies a times $a^{-1} = 1$

Zero power: If a is not zero, then a to the zero power is 1 so that the property of a^m times a^n equals $a^{(m+n)}$ continues to hold when $m = 0$ as well as for m and $n = 1, 2, 3, \dots$ When $a > 0$ it is possible to define a^x for any real x so that the above property holds. When $a = 0$, there are times when defining $a^0 = 1$ will cause problems, so this is usually not done.

Negative exponent: When $a > 0$, a^{-n} is defined to be $\frac{1}{a^n}$ so that the properties mentioned above continue to hold. When n is an integer, we can relax the restriction $a > 0$ to be a not equal to zero.

2. Find the prime factors of $6x^2 + xy - 9x - y^2 + 3y$.

Answer: Instead of "prime factors" of $6x^2 + xy - 9x - y^2 + 3y$ we would now ask that this be factored into linear factors. This can be done in a number of ways. One which is worth trying first is to factor $6x^2 + xy - y^2 = (3x - y)(2x + y)$ and factor the remaining terms as $-9x + 3y = (-3)(3x - y)$. Then factor the common factor $(3x - y)$ to get $(3x - y)(2x + y + 3)$.

3. Divide a by $\frac{x}{x+y} \times \frac{a}{x-y}$

Answer: a divided by the product of

$\frac{x}{x+y}$ times $\frac{a}{x-y}$ is a divided

by $\frac{(xa)}{(x^2 - y^2)}$ or a multiplied by

$\frac{(x^2 - y^2)}{xa}$ so it is $\frac{(x^2 - y^2)}{x}$

4. What is the greatest common divisor and the least common multiple of $x^2 - 4a^2$, $(x + 2a)^3$, and $(x - 2a)^3$?

Answer: The greatest common divisor of $(x - 2a)^3$ and $(x + 2a)^3$ is 1 so the greatest common divisor of the three function is 1. Since $x^2 - 4y = (x - 2a)(x + 2a)$, the least common

multiple of these three functions is the least common multiple of the two terms that are raised to the third power, so is $(x+2a)^3$ times $(x-2a)^3$ or $(x^2-4a^2)^3$

5. If B gives A \$5 of his money, A will have twice as much as B has left; but if A gives B \$5, then A will have but three-fourths as much as B will have. How much has each?

Answer: If B has b dollars and A has a dollars, when B gives A \$5 he has $b-5$ dollars and A has $a+5$ dollars. A then has twice as much money as B so as a first equation, we have

$$a+5=2(b-5)$$

As a second equation when A gives B \$5 and then has $\frac{3}{4}$ as much as B , we have

$$a-5=(\frac{3}{4})(b+5)$$

Multiply the second equation by 4 and use the distributive property

$$x(u+v)=xu+xv \text{ to get}$$

$$a+5=2b-10$$

$$4a-20=3b+15$$

Combine terms, subtracting 5 and $2b$ from both sides of the first equation, and adding 20 and subtracting $3b$ from both sides of the second equation. The equations are then

$$a-2b=-15$$

$$4a-3b=35$$

Multiply the first equation by -4 and add this to the second equation.

$$5b=95$$

Divide by 5 to get $b=19$, and then use the first equation to get $a=23$.

6. Extract the square root of

$$a^4 - a^3b + \frac{3a^2b^2}{4} - \frac{ab^3}{4} + \frac{b^4}{16}$$

Answer: To find the square root of

$$a^4 - a^3b + \frac{3a^2b^2}{4} - \frac{ab^3}{4} + \frac{b^4}{16}$$

one uses $(x+y)^2 = x^2 + 2xy + y^2$ to piece together what the square root has to be. First, $a^4 - a^3b$ can be made into a square by adding $\frac{a^2b^2}{4}$ so the first part of the square root

will involve $a^2 - \frac{ab}{2}$ since its square contains $a^4 - a^3b$

The remaining part of the square root comes from the square root of $\frac{b^4}{16}$, which is either $\frac{b^2}{4}$ or $\frac{-b^2}{4}$.

One can check that the first is correct so one square root of

$$\text{this polynomial is } a^2 - \frac{ab}{2} + \frac{b^2}{4}$$

There is a second square root which is the negative of this polynomial.

The fraction of polynomials, which are squares of other polynomials, is very small, so it was reasonable for this to have dropped out of school mathematics since the ideas involved in solving such problems can be taught with other problems which are more important. That is not true about taking square roots of positive numbers. Textbooks from the period of this exam and later into the early 20th century had nice explanations for how to numerically compute square roots. By the time I learned this around 1941 the reasons had disappeared, and now the procedures have also disappeared. Taking square roots should be returned to school mathematics; with reasons that are very important and applicable in other situations.

7. A merchant sold a piece of cloth for \$39 and gained as much percent as it cost him. How much did it cost him?

Answer: Let c dollars denote the cost of cloth to the merchant. $39 - c$ is the profit. The percentage gain is found by first computing $\frac{39-c}{c}$ and then multiplying by 100.

If this is equal to the original cost of the cloth, one needs to solve the equation $\frac{100(39-c)}{c} = c$ or $c^2 + 100c - 3900 = 0$.

The left-hand side can be factored as $(c-30)(c+130)$ so $c = \$30$ was the original cost. If you have trouble figuring out the factorization because there are many factors of 3900, it is possible to change the equation to reduce the size of the coefficients. Let $c = 10x$. The equation is then $100x^2 - 1000x - 3900 = 0$ and this is easier to factor as $(x-3)(x+13) = 0$.

8. Divide $7y^2$ by $\frac{10ax}{3by}$

Answer: $7y^2$ divided by $\frac{10ax}{3by}$ is $7y^2$ times $\frac{3by}{10ax}$ so the result is $\frac{21by^3}{10ax}$

9. Two houses standing on opposite sides of a street 84 feet in width, are respectively 67 and 54 feet in height. What length of rope will reach from the top of one house to the top of the other?

Answer: The distance between the houses would have been taken to be 84 feet, (no sidewalks between the house and the street), and the distance between the heights is 13 feet. One then uses the Pythagorean theorem to get the length of the rope (again ignoring the need to tie a knot in both ends); 84^2 is 7056 and 13^2 is 169. The sum of these two numbers is 7225 and this is 85^2 .