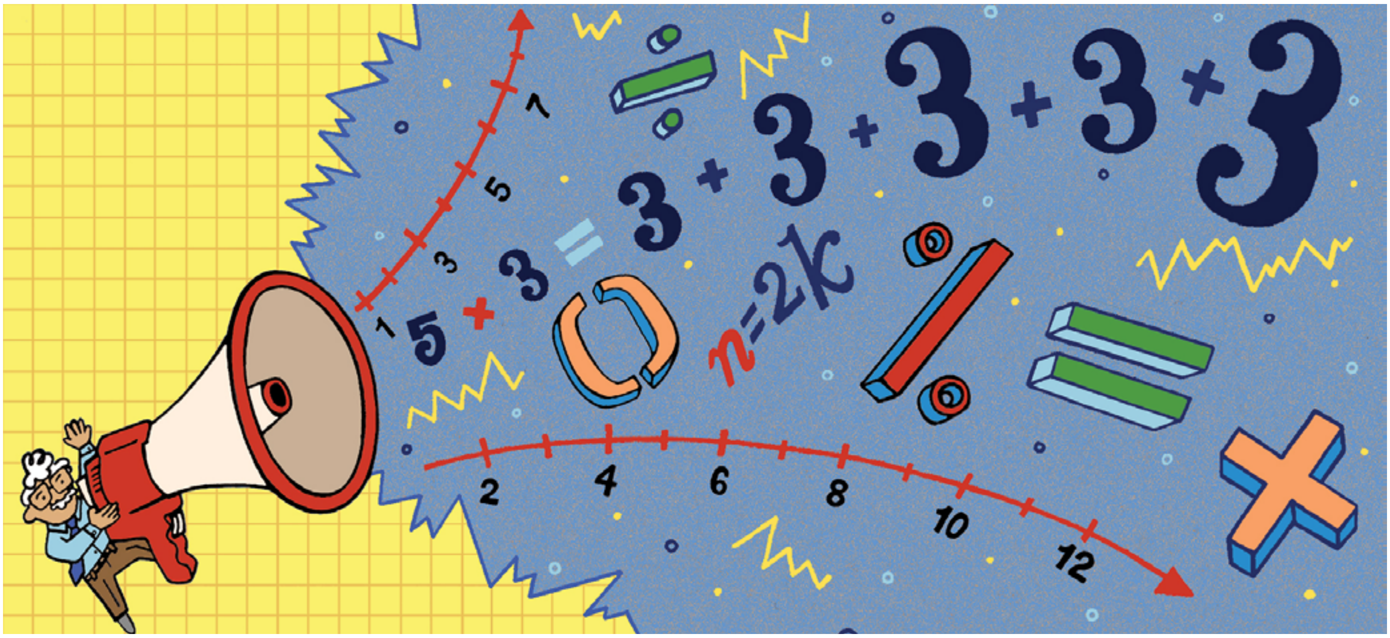
MATH MINUTE

Definitions in Mathematics

A 19th-Century Innovation That Supports Math Learning Today



ILLUSTRATIONS BY JONELL JOSHUA

By Jeremy F. Alm

Let's talk about definitions in mathematics. Unlike definitions you'd find in a dictionary, which are *descriptive*, definitions in mathematics are *prescriptive*: They declare that something is so, by fiat. When we mathematicians and mathematics educators make declarations like "A positive integer is called *semiprime* if it is the product of exactly two (not necessarily distinct) primes," we are endowing with meaning the term *semiprime*.

Such a definition is neither an empirical observation nor an edict from on high; it required building consensus throughout the mathematical community on its exact wording—a process that is often messy and fraught. Modern mathematical definitions are intended to be both useful and precise, not just decorations on classroom walls.* Let's look at each of those features, *useful* and *precise*, in turn.

Useful Definitions

When I say that definitions should be useful, I mean that they should serve as the foundation for reasoning in mathematics. Here is an example. An integer n is called *even* if it can be written as 2 times an integer—i.e., if $n = 2k$ for some integer k. Students often ask whether zero is even or odd. Apparently, at least in my experience as a mathematics professor, many students have been told that zero is neither even nor odd, which is incorrect. The definition of *even* will help us resolve this. Zero is even if $0 = 2k$ for some integer k. Can you think of such an integer k? My students usually can: use $k = 0$, so $0 = 2 \times 0$ shows that zero satisfies the definition of *even*.

To illustrate how a definition can be "correct" but less useful, let's consider an alternate definition of *even*: An integer is even if it is divisible by 2. This isn't wrong, but it is perhaps less clear whether or not zero is divisible by 2. It is, of course, but we would need to rely on a definition of "is divisible by" that itself is useful.

The idea that definitions are the "starting point" for our reasoning is often misunderstood. Definitions are the foundation for our deductive reasoning, so they are critical throughout our reasoning process, not just in the beginning. This point is illustrated rather humorously in many places on social media, with parents mocking some aspect of the Common Core. For example, in one YouTube video (available at go.aft.org/kwx), a parent is baffled that a child's response,

$$5 \times 3 = 5 + 5 + 5,$$

was marked incorrect. The correct answer is 3 + 3 + 3 + 3 + 3. (The video doesn't show what the prompt was, but it may have

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*For a lighthearted introduction to the importance of useful and precise definitions in a professional development or preservice course, see this activity for defining what a sandwich is (and is not): go.aft.org/32g.

Jeremy F. Alm is the director of the School of Mathematical and Statistical Sciences at Southern Illinois University in Carbondale, Illinois. His prior positions include math department head and, later, associate dean at Lamar University in Beaumont, Texas, and Illinois College's math department head.

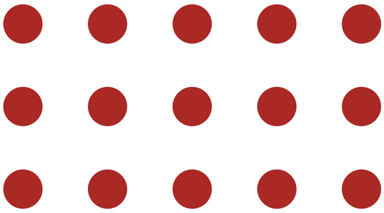
been something like, “By the definition of multiplication, what is 5 × 3?”) The parent’s argument is that both sums are equal to 15: “Does it really matter? It’s the same answer. They’re all right.” Of course, both are equal to 15. But how do we define *multiplication*?

A very common (pun intended) way to define 5 × 3 is “five groups of three,” which means 3 + 3 + 3 + 3 + 3. And in fact, this makes the response 5 + 5 + 5 incorrect. If the point of the question is to test whether students know the definition, then 5 + 5 + 5 must be considered incorrect. The fact that 3 + 3 + 3 + 3 + 3 = 5 + 5 + 5—and the fact that we can know this without calculating either sum—is one of the earliest interesting theorems in the elementary school math curriculum, namely that whole number multiplication is commutative.

To see that 5 × 3 = 3 × 5, first we use the definition of multiplication to rewrite each product as a sum:

$$3 + 3 + 3 + 3 + 3 = 5 + 5 + 5.$$

Now represent the sum 3 + 3 + 3 + 3 + 3 using a grid of dots, five columns of three dots:



By rotating the grid by 90 degrees, we can see that this very same grid also represents 5 + 5 + 5. This is a line of reasoning that elementary school students can follow.

Many of the comments on the YouTube video referenced commutativity of multiplication. Most blamed Common Core for all that ails us. None made any mention of the definition.

One thing that stood out from both the video and the comments was a focus on getting “the answer” in the quickest way possible. No attention was paid to justifying the reasoning involved in being able to explain the answer.

Precise Definitions

To explain what I mean by the claim that definitions should be precise, let’s consider the concept of percentage. A Google search of “define: percentage” returned exactly the sort of imprecision I am talking about:

- “a rate, number, or amount in each hundred”
- “any proportion or share in relation to a whole”
- “an amount, such as an allowance or commission, that is a proportion of a larger sum of money”

“A rate, number, or amount in each hundred”? That gets at the idea, but we can do better. Here’s how mathematicians define *percentage*: The number N percent means $\frac{N}{100}$.

How is this better? Let me illustrate with a question that I have asked of many people—and to which no nonmathematician has ever given a correct response: What percent of 17 percent is 31 percent?

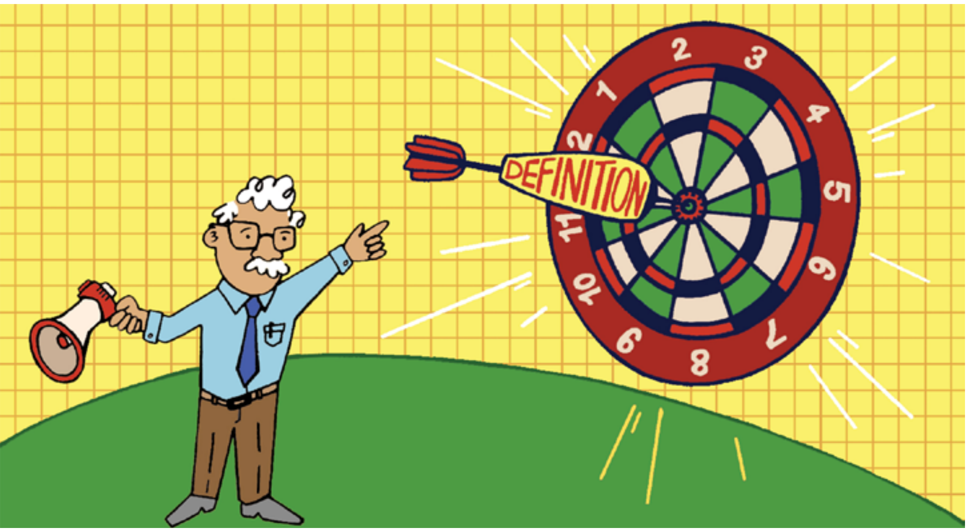
In my experience, nonmathematicians try to intuit their way to an answer, since they have been told that they need to “understand” percentages; if percentages are “amounts in each hundred,” then I guess there’s some work to be done to understand.

Instead, let’s translate using the definition, while acknowledging that “What” stands for an unknown, which we’ll call N; “of” means “multiply”; and “is” means “is equal to”:

$$\frac{N}{100} \times \frac{17}{100} = \frac{31}{100}.$$

Now the perplexing question is demystified. (The solution is $N = \frac{3100}{17}$, by the way.) Also notice some other things, like “200 percent of 5” is simply 2 × 5, and 100 percent is nothing more or less than the number 1.

When considering how precise definitions would be used in the classroom, a skeptic might counter that such definitions, and precise language in general, tend to increase cognitive load for students. This is almost certainly true when the definitions are introduced. One must put in effort to learn the precise language of mathematics. But once the definitions have been learned, they reduce cognitive load for students, just as knowing the definition of percentage makes figuring out “What percent of 17 percent is 31 percent?” easy. The broad point here is that the mathematics curriculum needs to reflect the logical structure of mathematics, which is reasoning based on definitions. That is the only way to prepare students for more advanced mathematical concepts. And so whatever pedagogy one adopts, it should be consistent with mathematical practice.



The mathematics curriculum must reflect the logical structure of mathematics: reasoning based on definitions.

Definitions as Innovations

If we go back prior to the 19th century, we find that mathematics operated differently from today. Definitions were often imprecise and intuitive. Like in the case of defining a fraction as "a part of a whole" (instead of a certain type of point on the number line, which is how mathematicians define it*), imprecise definitions can sound good and make people nod along, but as soon as you try to reason using them, their inadequacy is exposed.

Mathematicians in the 19th century noticed the power of precise definitions, and this led to a revolution in the way that mathematics was done. This revolution was remarkably successful and actually made mathematics much more accessible. A good definition is a tool designed to be used; in the hands of researchers and students alike, it contributes to success. Working with a well-formulated definition empowers the user, as illustrated with the definition of "percent" and as has been borne out in professional mathematics since the revolution.

Because the K–12 mathematics curriculum consists of mathematics that was developed prior to the 19th-century revolution, it has generally not reflected the changes that made mathematics more reliable and user-friendly.[1] Recently, the Common Core has brought definitions and, more importantly, *reasoning based on definitions* into the curriculum. But there is still a long legacy of argument by analogy, circular reasoning, and other practices that lead students to believe that mathematics isn't supposed to make sense; it's just something you must accept. (I will address these harmful practices in future articles.)

Implications for Teachers

In math education, there is tension in the supposed dichotomy between *procedural fluency* and *conceptual understanding*. But I am suggesting a different narrative: Definitions encapsulate essential features of examples of concepts.[2] Procedures are based on definitions, as are the explanations of why those procedures work.

In the classroom, definitions should follow examples that motivate them, and all reasoning should be based on definitions. So there should be no dichotomy. When people say that conceptual understanding should be emphasized before procedural fluency, I think they are recognizing that definitions should be motivated by discussions of the examples that inspire them, and that the procedures should follow the definitions.

Perhaps discussions of "concepts vs. procedures" could be resolved by attending to the role that useful, precise definitions play, and then we could put reasoning front and center in our classrooms instead of "answers." ■

Endnotes

1. For more on this revolution, see F. Quinn, "A Revolution in Mathematics? What Really Happened a Century Ago and Why It Matters Today," *Notices of the American Mathematical Society* 59, no. 1 (January 2012): 31–37.

2. For more on this, see D. Tall and S. Vinner, "Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity," *Educational Studies in Mathematics* 12 (1981): 151–69.

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*In an upper-level university mathematics course, one would use a more abstract definition. The number-line definition is equivalent and is appropriate for K–6.

Defining Odd

Given the definition of *even* above, one might be inclined to define an *odd integer* as one that cannot be written as 2 times an integer. This definition works, but could it be more useful and precise? Mathematicians define an odd integer to be one that can be written as one more than an even integer: n is odd if it can be written $n = 2k + 1$ for some integer k. But with either definition, there's some work to be done. If we start with "odd means not even," then we have to show that "odd implies *can be written as* $2k + 1$." If we start with "odd means *can be written as* $2k + 1$," then we need to show that every integer is either even or odd.

Let's do the latter. Let an integer n be called *odd* if there is some integer k such that $n = 2k + 1$. We will show that if an integer is not even, then it is odd. This might seem obvious, but really it's just familiar—so familiar it doesn't occur to us to question it. But for students who haven't known this "obvious" fact for years, it is not familiar and not obvious.

OK, let's get to work. Assume n is an integer that is *not* even. Then any even integer is either larger or smaller than n. Let $2k$ be the largest even integer smaller than n. (So, if n is 7, $2k$ would be 6.) Since $n > 2k$, there is some integer $x > 0$ such that $n = 2k + x$. Let's consider what x might be.

Can x be 2? No, it can't, because if $x = 2$, then $n = 2k + 2 = 2(k + 1)$, which would make n even, which by assumption it is not.

Can x be greater than 2? No, it can't, because if $x \geq 3$, then $n \geq 2k + 3 > 2k + 2$, and $2k + 2$ is an even integer less than n but *bigger* than $2k$. Since we picked $2k$ to be the largest even integer less than n, this can't be correct.

So, the only possibility left is that $x = 1$.

Therefore $n = 2k + 1$, and we have shown that if an integer isn't even, then it's odd. Hence every integer is either even or odd.

Note that we have relied on the definitions of *even* and *odd*, and we reasoned carefully at each step.

Would I recommend the above proof for a K–6 classroom? Of course not. But teachers who understand this reasoning can use their expertise to provide an age-appropriate explanation. What we don't want is to tell students "Every integer is either even or odd" without any explanation based on the definitions.

–J. F. A.