Why Is Elementary Math Scary?

Because Incoherence Abounds in Math Programs

By Jeremy F. Alm

"I just wanted to tell you I hate math." As a mathematics department head, I've heard this from too many prospective students at university recruiting events. So many students have had negative experiences in math class (myself included) that hating math class is a national pastime. But as someone who has for many years taught math to future teachers, I remain optimistic. I am convinced that students' (and teachers') negative attitudes are toward math *class*, not mathematics. This is because, as I try to illustrate in this article, many students have not experienced mathematics as my fellow mathematicians and I understand it: as a story well told.

Elementary mathematics, by which I mean mathematics that does not require algebra, is rich and beautiful. But it has a very bad PR department: Many of the textbooks that tell its story do it a grave injustice by not weaving a coherent narrative. When we watch a movie, following the plot requires that the narrative is coherent. We've all seen films that are impossible to follow because events occur for no apparent reason, without purpose or explanation. I claim that the narrative of elementary mathematics that has been pervasive in the United States from the late 20th century to the present is like this type of bad movie. It is unreasonable to expect students to follow the plot because the plot doesn't actually make sense.

I am a university professor, not a schoolteacher. What do I know about teaching K–12? Honestly, not much. I do not presume to tell teachers the finer points of pedagogy. What I do know is the difference between presentations of mathematics that make sense and those that do not. I have seen firsthand the harm done to young minds by having tried to survive in a learning environment in which the curriculum doesn't make sense. Like children that grow up in an unsafe environment, they can learn survival strategies that are not conducive to thriving at the next stage of their lives or education. We must fix the curriculum.

Incoherence: Creating a Fear of Fractions

I could give representative examples of incoherence from many of the widely used elementary math textbooks, but here I'm drawing from the 2015 edition of *Texas Go Math!* for fourth grade. This was the textbook used in my daughter's classroom at an elementary school in Texas during the 2018–19 school year.

1. Division of Whole Numbers

Suppose one wants to divide 29 by 5. There are several sensible things to write. The most immediate is 29 ÷ 5 = 29/5; since 29 is

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not a multiple of 5, the quotient is not an integer. Some might prefer the decimal 5.8. But one can make sense of division of integers without reference to fractions or decimals. Because the largest multiple of 5 less than or equal to 29 is $5 \times 5 = 25$, we can write

$$29 = 5 \times 5 + 4.$$

Students need to learn this concept, which we'll call *integer division-with-remainder*. Furthermore, they need to learn it early (see points 3 and 4 below).

Instead, in section 9.1 of *Texas Go Math!*, students are treated to this:[1]

$$29 \div 5 = 5 \text{ r4}.$$

There are two major problems with this. First, $29 \div 5$ is *a number*, and 5 r4 is not a number. As can be seen from the relation

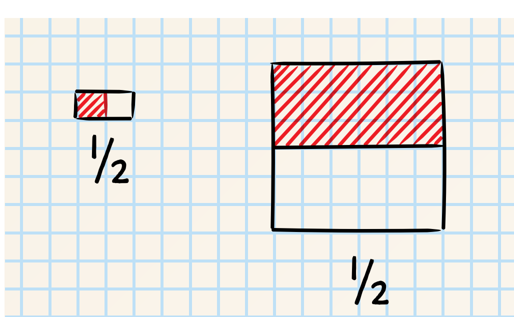
$$29 = 5 \times 5 + 4,$$

three numbers are required to write 29 as a multiple of 5, plus a remainder. The expression "5 r4" makes no reference to the divisor 5; as a result, it is meaningless.

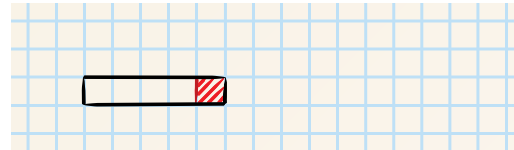
The second problem is that, following *Texas Go Math!*, we would also get $44 \div 8 = 5$ r4. Are we to conclude that $29 \div 5 = 44 \div 8$? We have two numbers, $44 \div 8$ and $29 \div 5$ both "equal" to 5 r4, but they are not equal to each other. This is not coherent.

2. Area-Model Fractions

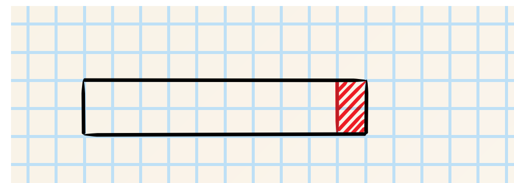
Texas Go Math! asks students to "use models to show equivalent fractions."[2] While there are plenty of acceptable ways to represent fractions visually, *Texas Go Math!* uses an area model that has no fixed unit. Rather, a fraction is represented by the *fraction of squares shaded*, no matter how many squares there are! For example, one can represent $\frac{1}{2}$ in the following two ways (and in many other ways):



Even worse, this approach misleads students about how large or small fractions are. Since one can represent $\frac{1}{5}$ as

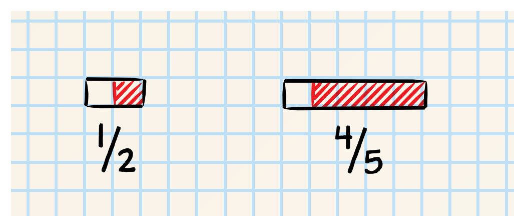


and $\frac{1}{10}$ as

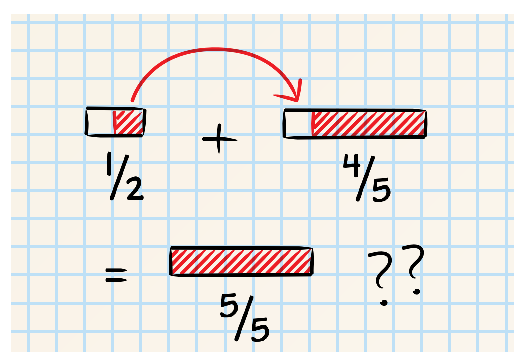


a student might reasonably conclude that $\frac{1}{10} > \frac{1}{5}$. After all, it has two boxes shaded, not just one.

Now consider the confusion that might be created when trying to compute $\frac{1}{2} + \frac{4}{5}$. Using this area model with no fixed unit, we use the simplest representations of the fractions 1/2 and 4/5:



Any sensible definition of addition must model our everyday notion of *combining quantities*, so one ought to be able to move the square from the representation of 1/2 to fill in the missing square in the representation of 4/5:



As a result, by using the model in a way that ought to make sense, a student may conclude that

$$\frac{1}{2} + \frac{4}{5} = \frac{5}{5} = 1.$$

This, of course, is not what *Texas Go Math!* concludes, but the reasoning seems consistent with its use of the area model.

A more acceptable way to proceed is to *fix a unit* for the entire discussion. For example, maybe the area of a 10-by-10 grid of squares is assigned to be the unit 1 for a section of the text. Then area comparisons can be made directly.

Here is an example that I have used in my courses with college students in both College Algebra and Math for Elementary Teachers.

> Suppose the unit 1 on the number line is the area of the following shaded region obtained from a division of a given square into 4 congruent rectangles (and therefore 4 parts of equal area):
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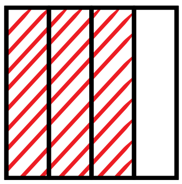
> 
>
> Write down the fraction of that unit representing the shaded area of the following division of the same square.
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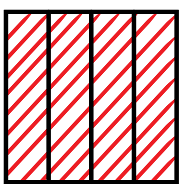
> 

Note that the unit is clearly defined. Also note that many of my students over the years have answered 1 to this question (instead of the correct 4/3).

3. Mixed Numbers and Improper Fractions

Section 3.6 in *Texas Go Math!* is especially problematic. First, students are expected to write

$$2\tfrac{3}{6} = 1 + 1 + \frac{3}{6}$$

when they haven't learned to add fractions yet.[3] Addition of fractions isn't introduced until Module 5. Second, it isn't made clear that

$$2\tfrac{3}{6}$$

is just shorthand for

$$2 + \frac{3}{6}.$$

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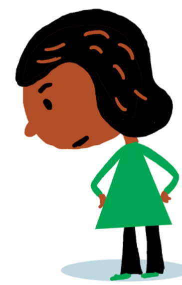
The correct way to show conversion from a mixed number to an improper fraction is as follows (assuming students already know how to add fractions):

$$\begin{aligned} 2\tfrac{3}{6} &= 2 + \frac{3}{6} \\ &= \frac{12}{6} + \frac{3}{6} \\ &= \frac{12+3}{6} \\ &= \frac{15}{6}. \end{aligned}$$



To go in the other direction, from improper fraction to mixed number, students should have learned integer division-with-remainder. Then we have

$$\begin{aligned} \frac{15}{6} &= \frac{2 \times 6 + 3}{6} \\ &= \frac{2 \times 6}{6} + \frac{3}{6} \\ &= 2 + \frac{3}{6} \\ &= 2\tfrac{3}{6}. \end{aligned}$$

4. Reducing Fractions

In section 3.3, students are encouraged to reduce $\frac{12}{16}$ as follows:[4]

$$\frac{12}{16} = \frac{12}{16} \div \frac{\square}{\square} = \frac{\square}{\square}$$

This is problematic for two reasons. First, fraction division hasn't even been introduced in the text yet. Presumably, students are supposed to write

$$\frac{12}{16} = \frac{12}{16} \div \frac{4}{4} = \frac{3}{4}.$$

Second, when they do learn fraction division, they will learn the invert-and-multiply rule, hence will be taught to write this:

$$\frac{12}{16} \div \frac{4}{4} = \frac{12}{16} \times \frac{4}{4} = \frac{48}{64},$$

which is totally unhelpful for reducing fractions.

The right way to teach this, at least initially, is to factorize the numerator and denominator into products of primes:

$$\frac{12}{16} = \frac{2 \times 2 \times 3}{2 \times 2 \times 2 \times 2} = \frac{3}{2 \times 2} = \frac{3}{4}.$$

When the numbers become difficult to factorize by inspection, as is the case with $\frac{391}{323}$, students who know integer division-with-remainder can easily be taught the Euclidean algorithm, which is simply repeated division-with-remainder:

$$\begin{aligned} 391 &= 1 \times 323 + 68 \\ 323 &= 4 \times 68 + 51 \\ 68 &= 1 \times 51 + 17 \\ 51 &= 3 \times 17 + 0. \end{aligned}$$

The last nonzero remainder—17 in this case—is the greatest common factor of 391 and 323, hence the reduced fraction is 23/19:

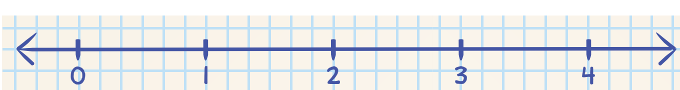
$$\frac{391}{323} = \frac{391 \div 17}{323 \div 17} = \frac{23}{19}.$$

Coherence: Fractions Are Numbers

The progression of "whole numbers, integers, fractions, rational numbers, real numbers" in the K–12 curriculum involves generalizing and extending what has come before to do arithmetic on larger and larger classes of numbers, each containing at least one of those that came before. Most difficulties for students seem to begin when moving from integers to fractions.

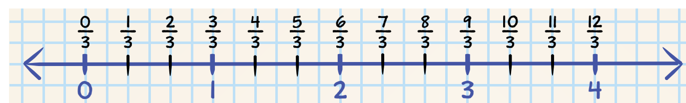
Let us consider addition: What for whole numbers and integers was counting on the number line, as in "5 + 2 is the number that is two steps to the right of 5 on the number line," becomes for fractions either a dubious analogy with apple pies or the incoherent area model shown above. However, there is another way: The number line provides a coherent common foundation for both integer and fraction arithmetic.

For integer arithmetic, the number line looks like this:



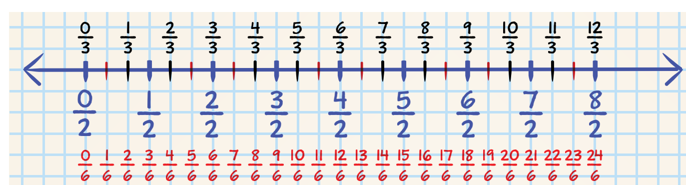
> For mathematics to be understandable and enjoyable, students need a coherent narrative to serve as the basis for reasoning.

When we introduce fractions, we simply add more markings. Let's consider thirds, for example:



Adding 5/3 and 2/3 is simply moving two steps to the right from 5/3 *on the sequence of thirds*. Thus, we have generalized integer addition by expanding the number of equally spaced sequences of points on the number line.

When students are ready to add fractions with different denominators, we must ask "How do we find a sequence that contains both 1/3 and 1/2?" The need for a *common denominator* becomes clear geometrically.



Fraction arithmetic is a modest generalization of integer arithmetic. When it is taught with a coherent narrative, it is not scary.

Parting Thoughts

In research-level mathematics, we often play around with toy examples until we grasp enough to build a theory that captures what's going on in the examples. The elementary school version of this is to let students explore specific cases to start to build some intuition, then provide a coherent narrative that can be the basis of reasoning (which is just the age-appropriate version of proof). It's great to have students explore with manipulatives, for example, for building early intuition and helping make the definitions seem sensible. But in order for mathematics to be understandable and enjoyable—beautiful, even—students still need a coherent narrative to serve as the basis for reasoning. Playing with egg cartons is great, but students' ability to follow the plot depends on the number line. ■

Endnotes

1. J. Dixon et al., *Texas Go Math!*, vol. 1 (Boston, MA: Houghton Mifflin Harcourt, 2015), 312, amazon.com/Houghton-Mifflin-Harcourt-Math-Texas/dp/0544061772/ref=sr_1_2.
2. Dixon et al., 75.
3. Dixon et al., 105.
4. Dixon et al., 91.