KNOWING AND TEACHING ELEMENTARY MATHEMATICS

By RICHARD ASKEY

THE TITLE of this article is also the title of a remarkable new book written by Liping Ma.[1] The basic format of the book is simple. Each of the first four chapters opens with a standard topic in elementary school mathematics, presented as a part of a situation that would arise naturally in a classroom. These scenarios are followed by extensive discussion by teachers regarding how they would handle each problem, and this discussion is interspersed with commentary by Liping Ma.

Here are the four scenarios:

Scenario 1: Subtraction with Regrouping

Let's spend some time thinking about one particular topic that you may work with when you teach: subtraction with regrouping. Look at these questions:

$$\begin{array}{r} 52 \\ -\ 25 \end{array} \qquad \begin{array}{r} 91 \\ -\ 79 \end{array}$$

How would you approach these problems if you were teaching second grade? What would you say pupils would need to understand or be able to do before they could start learning subtraction with regrouping?

* * *

Scenario 2: Multidigit Multiplication

Some sixth-grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate

$$\begin{array}{r} 123 \\ \times\ 645 \end{array}$$

the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They were doing this:

$$\begin{array}{r} 123 \\ \times\ 645 \\ \hline 615 \\ 492 \\ 738 \\ \hline 1845 \end{array}$$

instead of this:

$$\begin{array}{r} 123 \\ \times\ 645 \\ \hline 615 \\ 492 \\ 738 \\ \hline 79335 \end{array}$$

While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching sixth grade and you noticed that several of your students were doing this?

* * *

Scenario 3: Division by Fractions

People seem to have different approaches to solving problems involving division with fractions. How do you solve a problem like this one?

$$1\tfrac{3}{4} \div \tfrac{1}{2}$$

Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story problems to show the application of some particular piece of content. What would you say would be a good story or model for $1\tfrac{3}{4} \div \tfrac{1}{2}$?

* * *

Scenario 4: The Relationship Between Perimeter and Area

Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this

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picture to prove what she is doing:

4 cm		8 cm	
4 cm	☐	4 cm	▭

Perimeter = 16 cm   Perimeter = 24 cm
Area = 16 square cm   Area = 32 square cm
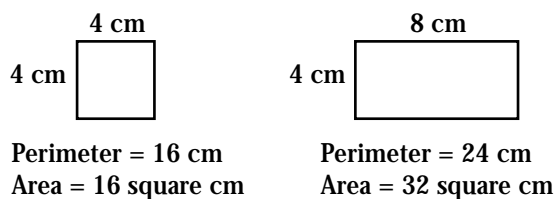
How would you respond to this student?

* * *

The 20- to 30-page discussions that follow each of these four problems are the richest examples I have encountered of teachers explaining what it means to really know and be able to teach elementary school mathematics. As the word "understanding" continues to be bandied about loosely in the debates over math education, this book provides a much-needed grounding. It disabuses people of the notion that elementary school mathematics is simple—or easy to teach. It cautions us, as Ma says in her conclusion, that "the key to reform…[is to] focus on substantive mathematics." And at the book's heart is the idea that student understanding is heavily dependent on teacher understanding. We can all learn from this book.

The problem that best illustrates the insights in this book is the one about the division of fractions. For that reason and because of space limitations, I will confine my comments in this article to that problem.

The teachers Ma interviewed composed numerous story problems to illustrate fractional division. They also explained the mathematical reasoning that underlies the calculation of division of fractions. And they provided mathematical proofs for their calculation procedures.

Before giving examples of story problems composed by the teachers Ma interviewed, it is worthwhile to give a general picture of different types of division problems, using whole numbers:

- 8 feet / 2 feet = 4 (measurement model)
- 8 feet / 2 = 4 feet (partitive model)
- 8 square feet / 2 feet = 4 feet (product and factors)

Now if we substitute fractions, using 1¾ in place of 8 and ½ in place of 2, these categories can be illustrated by the following examples:

- How many ½ foot lengths are there in something that is 1 and ¾ feet long?
- If half a length is 1 and ¾ feet, how long is the whole?
- If one side of a 1¾ square foot rectangle is ½ feet, how long is the other side?

Many other examples are given in Ma's book to represent this division problem. Here are two examples that use the measurement model:

Given that a team of workers construct ½ km of road each day, how many days will it take them to construct a road 1¾ km long?

Given that ½ apple will be a serving, how many servings can we get from 1¾ apples? (p. 73)

Many of the teachers favored the partitive model of division. Here are some of the story problems they composed based on that model:

Yesterday I rode a bicycle from town A to town B. I spent 1¾ hours for ½ of my journey; how much time did I take for the whole journey?

A factory that produces machine tools now uses 1¾ tons of steel to make one machine tool, ½ of what they used to use. How much steel did they used to use for producing one machine tool?

We want to know how much vegetable oil there is in a big bottle, but we only have a small scale. We draw ½ of the oil from the bottle, weigh it, and find that it is 1¾ kg. Can you tell me how much all the oil in the bottle originally weighed? (p. 79)

These are illuminating examples. They show the teachers' deep mathematical knowledge and their ability to represent mathematical problems to students. The latter has been called "pedagogical content knowledge."

It is important for students to learn both how to translate mathematical expressions into verbal problems and how to translate verbal problems into mathematical expressions that can be worked with. It is also important for students to understand how to do the calculation of division of fractions, and why this calculation works. Just telling students to "invert and multiply" is not enough. The following quotation from one of the teachers Ma interviewed starts with a brief statement about the relationship between division and multiplication. This statement provides a background for the story problem that follows.

Division is the inverse of multiplication. Multiplying by a fraction means that we know a number that represents a whole and want to find a number that represents a certain fraction of that. For example, given that we want to know what number represents ½ of 1¾, we multiply 1¾ by ½ and get ⅞. In other words, the whole is 1¾ and ½ of it is ⅞. In division by a fraction, on the other hand, the number that represents the whole becomes the unknown to be found. We know a fractional part of it and want to find the number that represents the whole. For example, if ½ of a jump rope is 1¾ meters, what is the length of the whole rope? We know that a part of the rope is 1¾ meters, and we also know that this part is ½ of the rope. When we divide the number of the part, 1¾ meters, by the corresponding fraction of the whole, ½, we get the number representing the whole, 3½ meters.… But I prefer not to use dividing by ½ to illustrate the meaning of division by fractions. Because one can easily see the answer without really doing division by fractions. If we say ⅘ of a jump rope is 1¾ meters, how long is the whole rope? The division operation will be more significant because then you can't see the answer immediately. The best way to calculate it is to divide 1¾ by ⅘ and get 2 3/16 meters. (p. 74)

This is a rich passage. The teacher begins by reminding her students that division is the inverse of multiplication. She then reviews what it means to multiply fractions, a topic that her students have already studied. Then building on their previous knowledge, the teacher offers an example that moves her class smoothly and logically to the division of fractions.

As the word 'understanding' continues to be bandied about loosely in the debates over math education, this book provides a much-needed grounding.
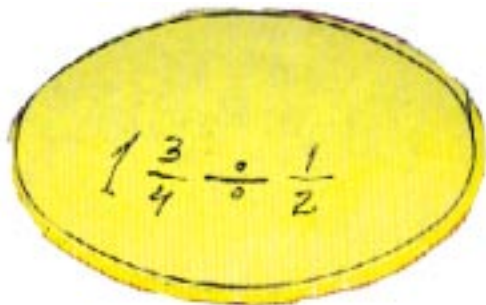
But this teacher is not content with the problem the interviewer gave her, $1\frac{3}{4} \div \frac{1}{2}$. She fears it will allow her students to "see the answer without really doing division by fractions." She substitutes a different problem—$1\frac{3}{4} \div \frac{4}{5}$— one that her students cannot easily visualize, thus forcing them deeper into the mathematics of the division of fractions.

This teacher is telling us something important about the level of knowledge needed if that knowledge is to be stable rather than fragile. If all that is expected of students is that they have a picture of how to deal with simple fractions like $\frac{1}{2}$ and $3\frac{1}{2}$, their knowledge will not be deep enough to build on. Likewise, if their knowledge is limited to the computational procedure without any idea why the procedure works, this is also not enough to build on. Students need both.

Like the teacher quoted above, many of the other teachers Liping Ma interviewed used the explanation that division is the inverse of multiplication. However, Ma points out that the teachers who used this explanation preferred the phrase "dividing by a number is equivalent to multiplying by its reciprocal." That is,

one can do division by multiplying by the reciprocal (or *inverse*) of the number being divided by. This is the mathematical reasoning that lies behind the "invert and multiply" computation.

Some of the teachers interviewed offered a formal mathematical proof to show why the algorithm for division of fractions works:

> OK, fifth-grade students know the rule of "maintaining the value of a quotient." That is, when we multiply both the dividend and the divisor with the same number, the quotient will remain unchanged. For example, dividing 10 by 2 the quotient is 5. Given that we multiply both 10 and 2 by a number, let's say 6, we will get 60 divided by 12, and the quotient will remain the same, 5. Now if both the dividend and the divisor are multiplied by the reciprocal of the divisor, the divisor will become 1. Since dividing by 1 does not change a number, it can be omitted. So the equation will become that of multiplying the dividend by the reciprocal of the divisor. Let me show you the procedure:
>
> $$\begin{aligned} 1\tfrac{3}{4} \div \tfrac{1}{2} &= (1\tfrac{3}{4} \times \tfrac{2}{1}) \div (\tfrac{1}{2} \times \tfrac{2}{1}) \\ &= (1\tfrac{3}{4} \times \tfrac{2}{1}) \div 1 \\ &= \ \ 1\tfrac{3}{4} \times \tfrac{2}{1} \\ &= \ \ 3\tfrac{1}{2} \end{aligned}$$
>
> With this procedure we can explain to students that this seemingly arbitrary algorithm is reasonable. (p. 60)

This is what Ma said of the teachers who offered proofs: "Their performance is mathematician-like in the sense that to convince someone of a truth one needs to prove it, not just assert it."

Many of the teachers Ma interviewed emphasized the necessity of thorough mastery of a topic before moving on to the next. In this instance, solid command of the multiplication of fractions was considered a "necessary basis" for approaching the division of fractions.

> The meaning of multiplication with fractions is particularly important because it is where the concepts of division by fractions are derived.... Given that our students understand very well that multiplying by a fraction means finding a fractional part of a unit, they will follow this logic to understand how the models of its inverse operation work. On the other hand, given that they do not have a clear idea of what multiplication with fractions means, concepts of division by a fraction will be arbitrary for them and very difficult to understand. Therefore, in order to let our students grasp the meaning of division by fractions, we should first of all devote significant time and effort when teaching multiplication with fractions to make sure students understand thoroughly the meaning of this operation.... Usually, my teaching of the meaning of division of fractions starts with a review of the meaning of multiplication with fractions. (p. 77)

This description shows an appreciation of how new knowledge is built on old knowledge. The insistence on mastery of a topic before moving on to the next stands in sharp contrast to the curriculum organization known as the "spiral curriculum." In the "spiral" approach to learning, mastery is not expected the first

(Continued on page 6)

Why does a negative × a negative = a positive?

(including how to explain it to your younger brother or sister)

For too many people, mathematics stopped making sense somewhere along the way. Either slowly or dramatically, they gave up on the field as hopelessly baffling and difficult, and they grew up to be adults who—confident that others share their experience—nonchalantly announce, "Math was just not for me" or "I was never good at it."

Usually the process is gradual, but for Ruth McNeill, the turning point was clearly defined. In an article in the Journal of Mathematical Behavior, *she described how it happened:[1]*

What did me in was the idea that a negative number times a negative number comes out to a positive number. This seemed (and still seems) inherently unlikely—counterintuitive, as mathematicians say. I wrestled with the idea for what I imagine to be several weeks, trying to get a sensible explanation from my teacher, my classmates, my parents, anybody. Whatever explanations they offered could not overcome my strong sense that multiplying intensifies something, and thus two negative numbers multiplied together should properly produce a *very* negative result. I have since been offered a moderately convincing explanation that features a film of a swimming pool being drained that gets run backwards through the projector. At the time, however, nothing convinced me. The most commonsense of all school subjects had abandoned common sense; I was indignant and baffled.

Meanwhile, the curriculum kept rolling on, and I could see that I couldn't stay behind, stuck on negative times negative. I would have to pay attention to the next topic, and the only practical course open to me was to pretend to agree that negative times negative equals positive. The book and the teacher and the general consensus of the algebra survivors of society were clearly more powerful than I was. I capitulated. I did the rest of algebra, and geometry, and trigonometry; I did them in the advanced sections, and I often had that nice sense of "aha!" when I could suddenly see how a proof was going to come out. Underneath, however, a kind of resentment and betrayal lurked, and I was not surprised or dismayed by any further foolishness my math teachers had up their sleeves.... Intellectually, I was disengaged, and when math was no longer required, I took German instead.

*Happily, Ruth McNeill's story doesn't end there. Thanks to some friendships she formed in college, her interest in math was rekindled. For most of our students, there is no rekindling. This is a tragedy, both for our students and for our country. Part of the reason students give up on math can be attributed to the poor quality of most of the math textbooks used in the United States. Many texts are written with the premise that if they end a problem with the words, "Explain your answer," they are engendering "understanding." However, because these texts do not give students what they would need to enable them to "explain," the books only add to students' mystifi-*cation and frustration.

Here is an example of how a widely acclaimed contemporary math series handles the topic that baffled Ruth McNeill: After a short set of problems dealing with patterns in multiplication of integers from 5 to 0 times (–4), the student is asked to continue the pattern to predict what (–1)(–4) is and then to give the next four equations in this pattern. There are then four problems, one of them being the product of two negative numbers. In the follow-up problems given next, there are four problems dealing with negative numbers, the last of which is the only one treating multiplication of negative numbers. This is how it reads: "When you add two negative numbers, you get a negative result. Is the same true when you multiply two negative numbers? Explain."

The suggested answer to the "explain" part is: "The product of two negative numbers is a positive." This is not an explanation, but a claim that the stated answer is correct.

Simply asking students to explain something isn't sufficient. They need to be taught enough so that they can explain. And they need to learn what an explanation is and when a statement is not an explanation.

The excerpt that follows is taken from a serious but lively volume entitled Algebra *by I.M.Gelfand and A. Shen, which was originally written to be used in a correspondence school that Gelfand had established. Contrast the inadequate treatment of the multiplication of negative numbers described above to the way Gelfand and Shen handle the topic.[2] Although their presentation would need to be fleshed out more if it's being presented to students for the first time, it provides us with a much better model for what "explain" might entail, offering as it does both an accessible explanation and a formal proof.*

—Richard Askey

The multiplication of negative numbers

To find how much three times five is, you add three numbers equal to five:

$$5 + 5 + 5 = 15.$$

The same explanation may be used for the product $1 \cdot 5$ if we agree that a sum having only one term is equal to this term. But it is evidently not applicable to the product $0 \cdot 5$ or $(–3) \cdot 5$: Can you imagine a sum with a zero or with minus three terms?

However, we may exchange the factors:

$$5 \cdot 0 = 0 + 0 + 0 + 0 + 0 = 0,$$

$$5 \cdot (–3) = (–3) + (–3) + (–3) + (–3) + (–3) = –15.$$

So if we want the product to be independent of the order of factors (as it was for positive numbers) we must agree that

$$0 \cdot 5 = 0, \quad (–3) \cdot 5 = –15.$$

Now let us consider the product $(-3) \cdot (-5)$. Is it equal to -15 or to $+15$? Both answers may have advocates. From one point of view, even one negative factor makes the product negative—so if both factors are negative the product has a very strong reason to be negative. From the other point of view, in the table

$3 \cdot 5 = +15$	$3 \cdot (-5) = -15$
$(-3) \cdot 5 = -15$	$(-3) \cdot (-5) = ?$

we already have two minuses and only one plus; so the "equal opportunities" policy requires one more plus. So what?

Of course, these "arguments" are not convincing to you. School education says very definitely that minus times minus is plus. But imagine that your small brother or sister asks you, "Why?" (Is it a caprice of the teacher, a law adopted by Congress, or a theorem that can be proved?) You may try to answer this question using the following example:

$3 \cdot 5 = 15$	Getting five dollars three times is getting fifteen dollars.
$3 \cdot (-5) = -15$	Paying a five-dollar penalty three times is a fifteen-dollar penalty.
$(-3) \cdot 5 = -15$	Not getting five dollars three times is not getting fifteen dollars.
$(-3) \cdot (-5) = 15$	Not paying a five-dollar penalty three times is getting fifteen dollars.

Another explanation. Let us write the numbers

$$1, 2, 3, 4, 5,...$$

and the same numbers multiplied by three:

$$3, 6, 9, 12, 15,...$$

Each number is bigger than the preceding one by three. Let us write the same numbers in the reverse order (starting, for example, with 5 and 15):

$$5, \ 4, \ 3, 2, 1$$
$$15, 12, 9, 6, 3$$

Now let us continue both sequences:

$$5, \ 4, 3, 2, 1, 0, -1, -2, -3, \ -4, \ -5,...$$
$$15, 12, 9, 6, 3, 0, -3, -6, -9, -12, -15,...$$

Here -15 is under -5, so $3 \cdot (-5) = -15$; plus times minus is minus.

Now repeat the same procedure multiplying 1, 2, 3, 4, 5,... by -3 (we know already that plus times minus is minus):

$$1, \ 2, \ 3, \ \ 4, \ \ \ 5$$
$$-3, -6, -9, -12, -15$$

Each number is three units less than the preceding one. Now write the same numbers in the reverse order:

$$5, \ \ 4, \ \ \ 3, \ \ 2, 1$$
$$-15, -12, \ -9, \ -6, -3$$

and continue:

$$5, \ \ 4, \ 3, \ 2, \ 1, \ \ 0, -1, -2, -3, -4, -5,...$$
$$-15, -12, -9, -6, -3, \ \ 0, \ \ 3, \ \ 6, \ \ 9, 12, 15,...$$

Now 15 is under -5; therefore $(-3) \cdot (-5) = 15$.

Probably this argument would be convincing for your younger brother or sister. But you have the right to ask: So what? Is it possible to *prove* that $(-3) \cdot (-5) = 15$?

Let us tell the whole truth now. Yes, it is possible to prove that $(-3) \cdot (-5)$ *must be* 15 *if* we want the usual properties of addition, subtraction, and multiplication that are true for positive numbers to remain true for any integers (including negative ones).

Here is the outline of this proof: Let us prove first that $3 \cdot (-5) = -15$. What is -15? It is a number opposite to 15, that is, a number that produces zero when added to 15. So we must prove that

$$3 \cdot (-5) + 15 = 0.$$

Indeed,

$$3 \cdot (-5) + 15 = 3 \cdot (-5) + 3 \cdot 5 = 3 \cdot (-5 + 5) = 3 \cdot 0 = 0.$$

(When taking 3 out of the parentheses we use the law $ab + ac = a(b + c)$ for $a = 3$, $b = -5$, $c = 5$; we assume that it is true for all numbers, including negative ones.) So $3 \cdot (-5) = -15$. (The careful reader will ask why $3 \cdot 0 = 0$. To tell you the truth, this step of the proof is omitted—as well as the whole discussion of what zero is.)

Now we are ready to prove that $(-3) \cdot (-5) = 15$. Let us start with

$$(-3) + 3 = 0$$

and multiply both sides of this equality by -5:

$$((-3) + 3) \cdot (-5) = 0 \cdot (-5) = 0.$$

Now removing the parentheses in the left-hand side we get

$$(-3) \cdot (-5) + 3 \cdot (-5) = 0,$$

that is, $(-3) \cdot (-5) + (-15) = 0$. Therefore, the number $(-3) \cdot (-5)$ is opposite to -15, that is, is equal to 15. (This argument also has gaps. We should prove first that $0 \cdot (-5) = 0$ and that there is only one number opposite to -15.)   □

[1] Ruth McNeill, "A Reflection on When I Loved Math and How I Stopped." *Journal of Mathematical Behavior,* vol. 7 (1988) pp. 45-50.

[2] *Algebra* by I. M. Gelfand and A. Shen. Birkhauser Boston (1995, Second Printing): Cambridge, Mass. © 1993 by I. M. Gelfand. Reprinted with permission.

time, and the same topics are revisited in two, three, and even four successive years.

In her discussion of the division of fractions, Ma mentions other methods of doing the calculation, including changing the problem to decimals, and dealing with numerators and denominators separately. The teachers who suggested these methods also noted that they were not always easier than the standard textbook method of multiplying by the reciprocal. The level of knowledge expected is illustrated by the following quotation:

> The teachers argued that not only should students know various ways of calculating a problem but they should also be able to evaluate these ways and to determine which would be the most reasonable to use. (p.64)

Throughout her book, Ma provides illustrations that help show how the topic under consideration fits into the larger picture of elementary mathematics. For example:

> The learning of mathematical concepts is not a unidirectional journey. Even though the concept of division by fractions is logically built on the previous learning of various concepts, it, in turn, plays a role in reinforcing and deepening that previous learning. For example, work on the meaning of division by fractions will intensify previous concepts of rational number multiplication. Similarly, by developing rational number versions of the two division models, one's original understanding of the two whole number models will become more comprehensive. (p. 76)

AS THE reader might have suspected from the measurement units used in some of the story problems, the teachers who were quoted are from a country that uses the metric system. The country is China, and these teachers live in Shanghai and neighboring areas of China. Liping Ma grew up in Shanghai until she was in the eighth grade, when China's "Cultural Revolution" sent her to the countryside for "re-education" by the peasants. In the poor rural village in South China where she was sent, the mostly illiterate villagers wanted their children to get an education. Ma was asked to teach, which she did for seven years, and later became elementary school superintendent for the county. Later she returned to Shanghai and started to read the classical works in the field of education. This eventually led her to Michigan State University (MSU) where she began working on a doctoral degree.

While at MSU, Liping Ma worked on a project run by Deborah Ball, which was a study to find out more about the mathematical knowledge of elementary school teachers in the United States. The four questions Ma used in her interviews with Chinese teachers were originally developed by Ball as part of the MSU study, and first used to interview U.S. teachers.[2] In her book, Ma draws on this database of U.S. teacher interviews as a point of comparison to the Chinese teachers.

The U.S. teachers fared poorly when compared to their Chinese counterparts. For the division of fractions problem discussed in this article, some of the U.S. teachers had difficulties with the calculations. None of them could adequately explain the mathematical reasoning embedded in the algorithm, provide appropriate real-world applications, or offer proofs.

It was not surprising to find that our elementary teachers' mathematical knowledge is not nearly as robust as that of the Chinese teachers. How could it be otherwise? Where could our teachers possibly have acquired the knowledge base that the Shanghai teachers demonstrated? Not from their own K-12 schooling, which focused mainly on developing a little skill on routine problems. Not from the math methods courses U.S. colleges offer, since these are light on math content. And—what may be surprising to many people—not even from the math courses they might have taken from a university mathematics department. At most colleges and universitites, there is a major disconnect between what is taught in these courses and the kind of math elementary school teachers need. As H. Wu has written: "There is an alarming *irrelevance* in the present preservice professional development in mathematics."[3]

A high school teacher who took a course from the well-known mathematician George Polya put it another way:

> The prospective teacher is badly treated both by the mathematics department and by the school of education. The mathematics department offers us tough steak which we cannot chew and the school of education vapid soup with no meat in it.[4]

It is not just the courses for high school math teachers that are problematic. Courses for prospective elementary school teachers, for example, frequently slight material dealing with fractions since whole number arithmetic is the main focus in our elementary schools. Middle school teachers frequently fall between the cracks. The material they will be teaching is not taught in detail to either prospective elementary school teachers or to prospective high school teachers; there are no courses specifically for middle school teachers.

If not from their pre-college education and not from their college education, where else might a U.S. teacher have acquired a deep understanding of mathematics? Perhaps from the textbooks and teachers' guides they use in their teaching. Liping Ma reports that Chinese teachers spend considerable time studying the textbooks:

Unfortunately, there are very few of our textbooks that a teacher would profit much from studying.
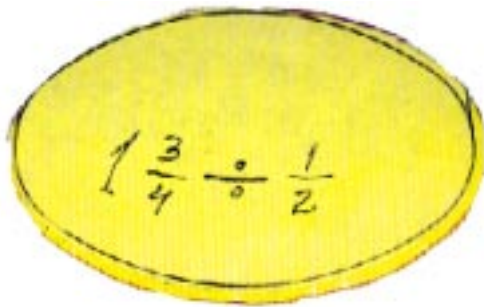
Teachers study textbooks very carefully; they investigate them individually and in groups, they talk about what textbooks mean, they do the problems together, and they have conversations about them. Teachers' manuals provide information about content and pedagogy, student thinking, and longitudinal coherence. (p. 149)

Unfortunately, there are very few of our textbooks that a teacher would profit much from studying.

The U.S. Department of Education has just announced the results of an exercise to identify "exemplary" and "promising" texts. *Connected Mathematics,* a series for grades 6-8, is one the department has deemed exemplary. I do not understand why it deserves that rating. I am quite familiar with this series, as I reviewed it as part of a textbook adoption process. Regarding fractions, for example, *Connected Math* has some material on the addition and subtraction of fractions, but nothing as systematic as described by the Chinese teachers interviewed by Ma. There is less on multiplication of fractions, and *nothing* on the division of fractions. If our students go through grade 8 without having studied the division of fractions, where are our future primary teachers going to learn this? The criteria used by the Department of Education review

should be rewritten now that Liping Ma's book has provided us with a model of what school mathematics should look like.

Another recent development that leaves me less than encouraged is the way fractions are addressed in the draft of the revised K-12 mathematics standards released last year by the National Council of Teachers of Mathematics (*Principles and Standards for School Mathematics: Discussion Draft*[5]). Most of the work on fractions has been put in the grades 6 to 8 band. Students are to "develop a deep understanding of rational number concepts and reasonable proficiency in rational-number computation." It is the adjective "reasonable" that bothers me. *Proficiency* should be the goal. It is hard to imagine the Chinese teachers that Ma interviewed settling for "reasonable" proficiency with fractions for their students. These lower expectations show in every international comparison.

Furthermore, the only problem used to illustrate division of fractions in NCTM's draft revision is how many pieces of ribbon $\frac{3}{4}$ yards long can be cut from $4\frac{1}{2}$ yards of ribbon. The text continues with: "The image is of repeatedly cutting off $\frac{3}{4}$ of a yard of ribbon. Having students work with concrete objects or drawings is helpful as students develop and deepen their understanding of operations." It seems that we are back again to simple fractions and concrete objects that students can visualize. Contrast this with what Liping Ma observed:

> The concept of fractions as well as the operations with fractions taught in China and the U.S. seem different. U.S. teachers tend to deal with "real" and "concrete" wholes (usually circular or rectangular shapes) and their fractions. Although Chinese teachers also use these shapes when they introduce the concept of a fraction, when they teach operations with fractions they tend to use "abstract" and "invisible" wholes (e.g., the length of a particular stretch of road, the length of time it takes to complete a task...). (p. 76)

THE LAST three chapters in Liping Ma's book deal with *when* the Chinese teachers acquired the knowledge they showed, and a description of what Ma calls "Profound Understanding of Fundamental Mathematics," or PUFM. Here is part of her description:

> A teacher with PUFM is aware of the "simple but powerful" basic ideas of mathematics and tends to revisit and reinforce them. He or she has a fundamental understanding of the whole elementary mathematics curriculum, thus is ready to exploit an opportunity to review concepts that students have previously studied or to lay the groundwork for a concept to be studied later. (p. 124)

From their pre-collegiate studies, the Chinese teachers Ma interviewed had a firm base of knowledge on which to build. However, PUFM did not come directly from their studies in school, *but from the work they did as teachers.* These teachers did not specialize in mathematics in "normal" school, which is what their teacher preparation schools are called. But after they started teaching, most of them taught only mathematics or mathematics and one other subject. This allowed them to specialize in ways that few of our ele-

mentary school teachers can. Quite a few regularly changed the level at which they taught. They might go through a cycle of three grades, then repeat the same cycle, or change and teach a different age group. This allows them to see the development of mathematics from the perspective of a teacher, something too few of our elementary school teachers are able to do.

Recently, the Learning First Alliance—an organization composed of many of the major national education organizations—recommended that beginning in the fifth grade, every student should be taught by a mathematics specialist.[6] This is a hopeful development, and for many teachers it would mean unburdening themselves from something they now find difficult and unpleasant.

There is more we can do. Our teachers need good textbooks. They need much better teachers' manuals. As noted before, our college math courses for future teachers at all levels need to be improved. And just ask any teacher who has sat through mindless "workshops" whether our in-service "professional development" isn't long overdue for major overhaul.

Teachers also need *time* to prepare their lessons and further their study of mathematics. Recall Ma's comments that it is *during* their teaching careers that Chinese teachers perfect their knowledge of mathematics. Listen to this Shanghai teacher describe his class preparation:

> I always spend more time on preparing a class than on teaching, sometimes three, even four, times the latter. I spend the time in studying the teaching materials: What is it that I am going to teach in this lesson? How should I introduce the topic? What concepts or skills have the students learned that I should draw on? Is it a key piece on which other pieces of knowledge will build, or is it built on other knowledge? If it is a key piece of knowledge, how can I teach it so students can grasp it solidly enough to support their later learning? If it is not a key piece, what is the concept or the procedure it is built on? How am I going to pull out that knowledge and make sure my students are aware of it and the relation between the old knowledge and the new topic? What kind of review will my students need? How should I present the topic step-by-step? How will students respond after I raise a certain question? Where should I explain it at length, and where should I leave it to students to learn it by themselves? What are the topics that the students will learn which are built directly or indirectly on this topic? How can my lesson set a basis for their learning of the next topic, and for related topics that they will learn in their future? What do I expect the advanced students to learn from the lesson? What do I expect the slow students to learn? How can I reach these goals? etc. In a word, one thing is to study whom you are teaching, the other thing is to study the knowledge you are teaching. If you can interweave the two things together nicely, you will succeed. We think about these two things over and over in studying teaching materials. Believe me, it seems to be simple when I talk about it, but when you really do it, it is very complicated, subtle, and takes a lot of time. It is easy to be an elementary school teacher, but it is difficult to be a good elementary school teacher. (p. 135)

LIPING MA'S book provides a start to what I hope will be a continuing study of fundamental mathematics and the connections between different parts of it. We need many more commentaries on the teaching of mathematics like those contained in Ma's book. We also need more detailed lesson plans, as are frequently provided in Japan.[7] There are a few places where one can read comments by U.S. teachers or by mathematics education researchers. However, these comments almost all deal with the initial steps of an idea, which typically means using pictures or manipulatives to try to get across the basic concept. Almost never is there elaboration of what should be done *next*, to help develop a deeper view of the subject, which will be necessary for later work.

And elementary school mathematics *is* much deeper, more profound, than almost everyone has thought it to be. As Ma comments, toward the end of her book:

> In the United States, it is widely accepted that elementary mathematics is "basic," superficial, and commonly understood. The data in this book explode this myth. Elementary mathematics is not superficial at all, and any one who teaches it has to study it hard in order to understand it in a comprehensive way. (p. 146)

But, she concludes:

> The factors that support Chinese teachers' development of their mathematical knowledge are not present in the United States. Even worse, conditions in the United States militate against the development of elementary teachers' mathematical knowledge.... (p.xxv)

This must change. We cannot continue to abandon teachers at every critical stage of their development and then send them into the classroom with a mandate to "teach for understanding." This is dishonest and irresponsible. As things stand now, we are asking teachers to do the impossible. They and the students they teach deserve better. □

REFERENCES

[1]Liping Ma, *Knowing and Teaching Elementary Mathematics*, Lawrence Erlbaum Associates, Mahwah, NJ, 1999.

[2]Deborah Ball, "Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education." Unpublished Ph.D. thesis, Michigan State University, 1988 and published papers on this topic.

[3]H.Wu, "Preservice Professional Development of Mathematics Teachers." Unpublished manuscript (1999). Available at www.math.berkeley.edu/~wu

[4]George Polya, "Ten Commandments for Teachers," *Journal of Education of the Faculty and College of Education of the University of British Columbia*, (3) 1959, pp. 61-69.

[5]National Council of Teachers of Mathematics, *Principles and Standards for School Mathematics: Discussion Draft*, NCTM, Reston, Va., 1998.

[6]Learning First Alliance, *An Action Plan for Changing School Mathematics*, Washington, D.C., 1998.

[7]Alice Gill and Liz McPike, "What We Can Learn from Japanese Teachers' Manuals," *American Educator*, vol. 19, no. 1, Spring 1995, pp. 14-24.