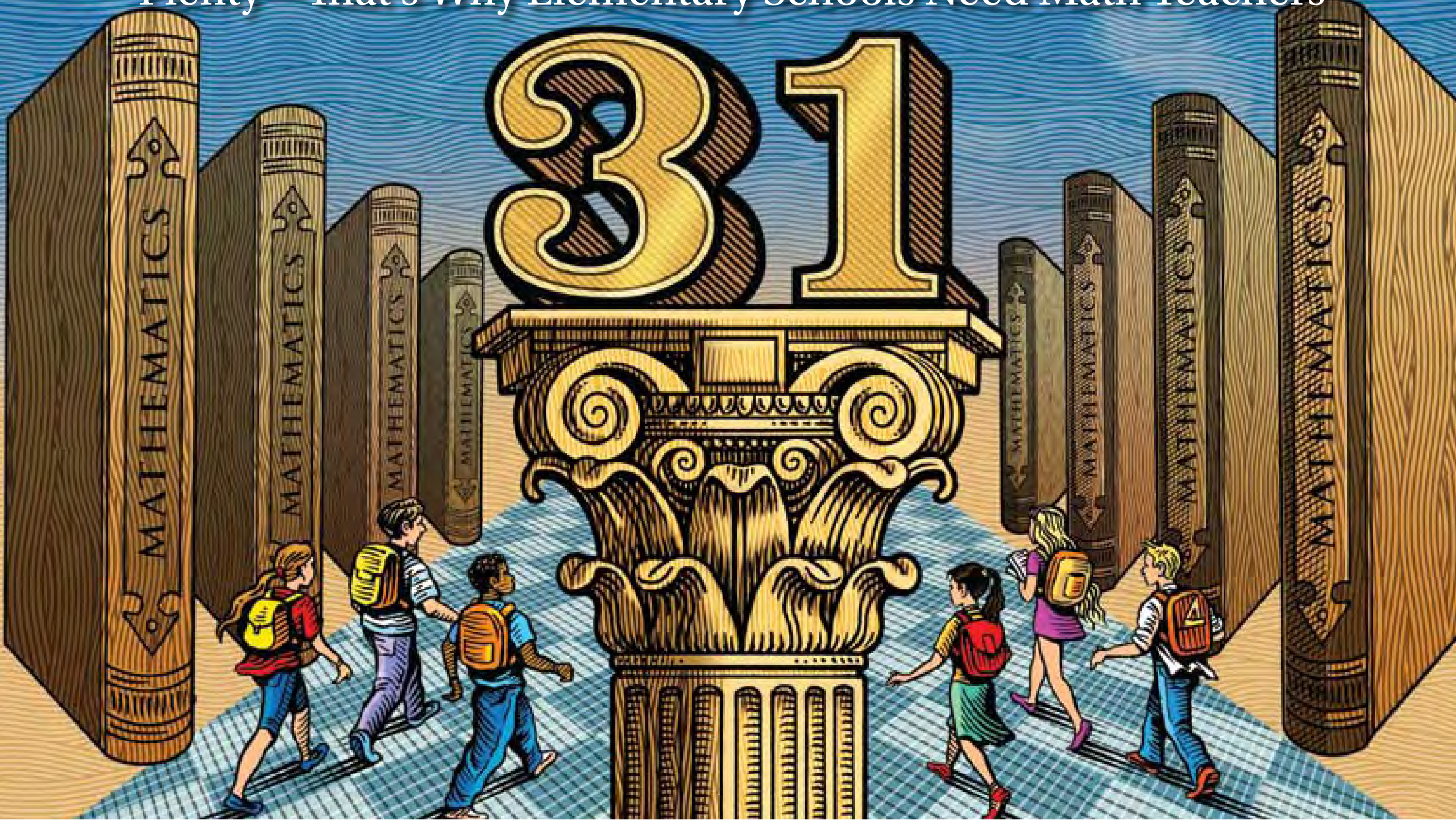
What's Sophisticated about Elementary Mathematics?

Plenty—That's Why Elementary Schools Need Math Teachers



ILLUSTRATED BY ROLAND SARKANY

BY HUNG-HSI WU

Some 13 years ago, when the idea of creating a cadre of mathematics teachers for the upper elementary grades (who, like their counterparts in higher grades, would teach only mathematics) first made its way to the halls of the California legislature, the idea was, well, pooh-poohed. One legislator said something like: "All you have to do is add, subtract, multiply, and divide. How hard is that?"

The fact is, there's a lot more to teaching math than teaching how to do calculations. And getting children to understand important ideas like place value and fractions is hard indeed.

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As a mathematician who has spent the past 16 years trying to improve math education—including delivering intensive professional development sessions to elementary-grades teachers—I am an advocate for having math instruction delivered by math teachers as early as possible, starting no later than fourth grade.* But I also understand that until you appreciate the importance and complexity of elementary mathematics, it will not be apparent why such math teachers are necessary.

In this article, I address two "simple" topics to give you an idea of the advanced content knowledge that is needed to teach math effectively. Our first topic—adding two whole numbers—is especially easy. The difficulty here is mostly in motivating and engaging students so that they come to understand the standard addition algorithm and, as a result, develop a deeper appreciation of place value (which is an absolutely critical topic in elementary

*There have been calls for math teachers in the education literature, among them the National Research Council's *Adding It Up* (see pages 397–398, available at www.nap.edu/catalog.php?record_id=9822#toc) and the National Mathematics Advisory Panel's *Foundations for Success* (see Recommendation 20 on page xxii, available at www.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf).

math). This discussion of addition may not convince you that math teachers are a necessity in the first through third grades, but it will give you a deeper appreciation of the important mathematical foundation that is being laid in the early grades.

Our second topic—division of fractions—is substantially harder, though it's still part of the elementary mathematics content as it should be taught in fifth and sixth grades. This is a topic that, in my experience, many adults struggle with. My goal here is twofold: (1) to show you that elementary math can be quite sophisticated, and (2) to deepen your knowledge of division and fractions. Along the way, I think it will become apparent why mathematicians consider facility with fractions essential to, and excellent preparation for, algebra. By the end, I hope you will join me in calling for the creation of a cadre of teachers who specialize in the teaching of mathematics in grades 4–6. For simplicity, we will refer to them as *math teachers*, to distinguish them from *elementary teachers* who are asked to teach all subjects.

Adding Whole Numbers

Consider the seemingly mundane skill of adding two whole numbers. Take, for example, the following.

$$\begin{array}{r} 45 \\ \underline{+\,31} \\ 76 \end{array}$$

Nothing could be simpler. This is usually a second-grade lesson, with practice continuing in the third grade. But if you were the teacher, how would you convince your students that this is worth learning? Too often, children are given the impression that they must learn certain mathematical skills because the teacher tells them they must. So they go through the motions with little personal involvement. This easily leads to learning by rote. How, then, can we avoid this pitfall for the case at hand? One way is to teach them what it means to add numbers, why it is worth knowing, why it is hard if it is not done right, and finally, why it can be fun if they *learn how to add the right way*.

All this can be accomplished if you begin your lesson with a story, like this: Alan has saved 45 pennies and Beth has saved 31. They want to buy a small package of stickers that costs 75 cents, and they must find out if they have enough money together. To act this out, you can show children two bags of pennies, one bag containing 45 and the other 31. Now dump them on the mat and explain that they have to count how many there are in this pile. Chances are, they will mess up as they count. Let them mess up before telling them there is an easier way. Go back to the bags of 45 and 31, and explain to them that it is enough to begin with 45 and continue to count the pennies in the bag of 31. In other words, *to find out how many are in 45 and 31 together, start with 45 and just go 31 more steps; the number we land on is the answer*. To show them that making these steps corresponds exactly to counting, do a simple case with them. If there are 3 pennies in the smaller bag instead of 31, then going 3 steps from 45 lands at 48 because

$$45 \quad \rightarrow \quad 46 \quad \rightarrow \quad 47 \quad \rightarrow \quad 48.$$

So 48 is the total number of pennies in the two bags of 45 and 3. Now ask them to count like this for 45 and 31; chances are, most of them will find this a bit easier but many will still mess up. You can help them get to 76, but they probably will get frustrated. That is good: here is something they want to learn, but they find it is not so easy.

Then you get to play the magician. Tell them that what they are doing is called "adding numbers." In this case, they are adding 31 to 45, written as 45 + 31 (teach them to write addition horizontally as well as vertically from the beginning), and what it means is that it is the number they get by starting with 45 and counting 31 more steps. Show them they do not have to count so strenuously to get the answer to 45 + 31 because they can do two simple additions instead, one being 4 + 3 and the other 5 + 1, and these give the two digits of the correct answer 76.

> All whole-number computations are nothing but a sequence of single-digit computations artfully put together. This is the kind of thinking students will need to succeed in algebra and advanced mathematics.

You can demonstrate this effectively by collecting the 45 pennies and putting them into bags of 10; there will be 4 such bags with 5 stragglers. Do the same with the other 31 pennies. Then place these bags and stragglers on the mat again, and ask them how many pennies there are. It won't take long for them to figure out that there are 4 + 3 bags of 10, and 5 + 1 stragglers.

They will figure out that 7 bags of 10 together with 6 stragglers total 76 again. Now ask them to compare counting the bags and stragglers with the magic you performed just a minute ago. If they don't see the connection (and some won't), patiently explain it to them. Of course, this is the time to review place value. (To better understand place value, and to prepare for the occasional advanced student, see the sidebar on page 9.) Then, you can use place value to explain that when they add the 4 bags of 10 to the 3 bags of 10, they are actually adding 40 and 30.

$$\begin{array}{r} 45 \\ \underline{+\,31} \\ 76 \end{array} \;\rightarrow\; \begin{array}{r} 40+5 \\ \underline{+\,30+1} \\ ?\quad ? \end{array} \;\rightarrow\; \begin{array}{r} 40+5 \\ \underline{+\,30+1} \\ 70+6 \end{array} \;\rightarrow\; \begin{array}{r} 45 \\ \underline{+\,31} \\ 76 \end{array}$$

Now, they will listen more carefully to your incantations of place value because you have given them more incentive to learn about this important topic.

As mentioned above, addition of whole numbers is done mainly in grades 2 and 3. Often, the addition algorithm is taught by rote, but some teachers do manage to explain it in terms of place value, as we have just done. Many educators believe that the real difficulty of this algorithm arises when "carrying" is necessary, but conceptually, carrying is just a sidelight, a little wrinkle on the fabric. The key idea is contained in the case of adding without carrying. If we succeed in getting students to *thoroughly understand* addition without carrying, then they will be in an excellent position to handle carrying too. (However, in my experience, the standard textbooks and teaching in most second- or third-grade classrooms focus on carrying before students are ready, and that is a pity.)

Understanding the addition algorithm in terms of place value—for example, that 45 + 31 is 40 + 30 and 5 + 1—is appropriate for beginners, but it cannot stop there. The essence of the addition algorithm, like all standard algorithms, lies in the abstract understanding that *the arithmetic computations with whole numbers, no matter how large, can all be reduced to computations with single-digit numbers*. (For more on this, see the sidebar on page 10.) In other words, students' ultimate understanding of these algorithms must transcend place value to arrive at the recognition that all whole-number computations are nothing but a sequence of single-digit computations artfully put together. This is the kind of thinking students will need to succeed in algebra and advanced mathematics. More precisely, students should get to the point of recognizing that 45 + 31 is no more than the combination of two single-digit computations, 4 + 3 and 5 + 1. Whether the 4 stands for 40 or 40,000 and the 3 stands for 30 or 30,000 is completely irrelevant.

To drive home this point, consider the following two addition problems.

$$\begin{array}{r} 45 \\ \underline{+\,31} \\ 76 \end{array} \qquad\qquad \begin{array}{r} \boldsymbol{45}723 \\ \underline{+\,\boldsymbol{31}251} \\ \boldsymbol{76}974 \end{array}$$

The problem on the left is the one we have been working with, and parts of the problem on the right are tantalizingly similar, except that the 4 and 5 in the first row are no longer 40 and 5 but 40,000 and 5,000, respectively. Similarly, the 3 and 1 in the second row are not 30 and 1 but 30,000 and 1,000, respectively. Yet, do the changes in the place values of these four single-digit numbers (4, 5, 3, and 1) change the addition? Not at all, because the result is still the same two digits, 7 and 6, and that is the point.

We are now able to directly address the main concern of this article, which is the need for math teachers at least starting in grade 4. In grade 4, the multiplication algorithm has to be explained. A teacher knowledgeable in mathematics would know that this is the time to cast a backward glance at the addition algorithm to make sure students finally grasp a real understanding of what this algorithm is all about: just a sequence of single-digit computations. Why is this knowledge so critical at this point? Because it leads seamlessly to the explanation of why students must memorize the multiplication table (of single-digit numbers) to automaticity before they do multidigit multiplication: in the same way that knowing how to add single-digit numbers enables them to add any two numbers, no matter how large, knowing how to multiply single-digit numbers enables them to multiply any two numbers, no matter how large. We want students to be exposed, as early as possible, to the idea that beyond the nuts and bolts of mathematics, there are unifying undercurrents that connect disparate pieces.

Let us go a step further to make explicit the role of single-digit computations in the additions of 45 + 31 and 45723 + 31251. If students have been given the proper foundation in second grade, then in fourth grade, a math teacher will be able to give the following explanation.

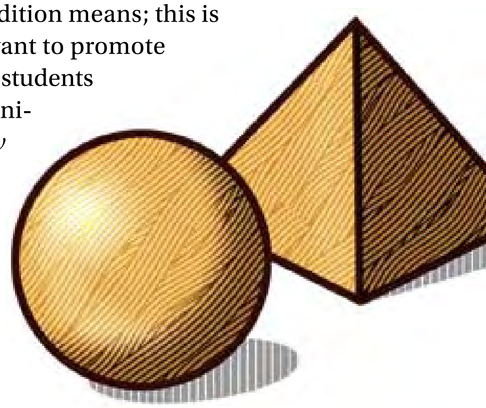
$$\begin{aligned} 45 + 31 &= (4 \times 10) + 5 + (3 \times 10) + 1 \\ &= (4 \times 10) + (3 \times 10) + 5 + 1 \\ &= \boldsymbol{(4 + 3) \times 10 + (5 + 1)} \end{aligned}$$

In the last equality, we used the distributive law—i.e., (b + c)a = ba + ca—to rewrite (4 × 10) + (3 × 10) as (4 + 3) × 10. For 45723 + 31251, we will focus only on 45 and 31 to enhance clarity. We have, then, the following.

$$\begin{aligned} 45723 + 31251 &= (4 \times 10000) + (5 \times 1000) + \cdots \\ &\quad + (3 \times 10000) + (1 \times 1000) + \cdots \\ &= (4 \times 10000) + (3 \times 10000) + \cdots \\ &\quad + (5 \times 1000) + (1 \times 1000) + \cdots \\ &= \boldsymbol{(4 + 3) \times 10000 + (5 + 1) \times 1000 + \cdots} \end{aligned}$$

Again, the last equality makes use of the distributive law. If we compare the two expressions (4 + 3) × 10 + (5 + 1) and (4 + 3) × 10000 + (5 + 1) × 1000, we see clearly that the *same* single-digit additions (4 + 3) and (5 + 1) are in both of them, and that the difference between these expressions lies merely in whether these single-digit sums are multiplied by 10 or 1,000 or 10,000, the place values of the respective digits. This clearly illustrates the primacy of single-digit computations in the addition algorithm.

Returning to our original second-grade lesson of 45 + 31, let's review what you have accomplished. You have shown students what addition means; this is important because we want to promote the good practice among students that through precise definitions, they *get to know what they will do before doing it*. Then you made them want to learn it, and made them realize that the most obvious method (counting) is not the easiest. Best of all, you opened their eyes to the magic of learning: acquiring the power of making something complicated much simpler. Instead of tedious, error-prone counting, you used the concept of place value to introduce the idea of breaking up a task digit by digit and adding only two single-digit numbers in succession. A couple of years later, the fourth-grade math teacher will have the opportunity to explain and make explicit the idea that to add any whole numbers, no matter how large, all the children need to do is add single-digit numbers.



The main goal of the elementary mathematics curriculum is to provide children with a good foundation for mathematics. In this context, the addition algorithm, when taught as described above in grades 2–4, serves as a splendid introduction. It teaches children an important skill in mathematics: *if possible, always break up a complicated task into a sequence of simple ones*. This is why we do not look at 45 and 31, but only 4 and 3, and 5 and 1.

Of course, they will encounter somewhere down the road something like 45 + 37, but they will be in a position to understand that the carrying step is actually adding a 1 to the 10s column. Despite how it is presented in most U.S. textbooks, carrying is *not* the main idea of the addition algorithm. The main idea is to break up any addition into the additions of single-digit numbers and then, drawing on our understanding of place value, put these simple computations together to get the final answer. If you can make your students understand that, you are doing fantastically well as a teacher, because you have taught them important mathematics. They now have an important skill *and* know the reasoning behind it—and they will have used both to deepen their appreciation of place value.

Dividing Fractions

I've had plenty of encounters with well-educated adults who can't divide fractions without a calculator, or who can, but have no idea why the old rule "invert and multiply" works. With that in mind, I'll break this topic into three parts: we'll review division, then fractions, and finally the division of fractions. Along the way, the answer to our larger question—what's sophisticated about elementary mathematics?—will become apparent, as will the ways in which mastering fractions prepares students for algebra.

Let's begin with the division of whole numbers, which would

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normally be taught in third grade. What does 24/6 = 4 mean? In the primary grades, we teach two meanings of division of whole numbers: partitive division* and measurement division. For brevity, let us concentrate only on measurement division, in which the meaning of 24/6 = 4 is that by separating 24 into equal groups of 6, we find that there are 4 groups in all. So the *quotient* 4 tells how many groups of 6s there are in 24.

By fifth grade, students should be ready to apply their understanding of measurement division to a more symbolic format. This will prepare them for the division of fractions, for which the idea of "dividing into equal groups" often is not very helpful in calculating answers. (For example, the division of $\frac{1}{7}$ by $\frac{1}{2}$ does not lend itself to any easy interpretation of dividing $\frac{1}{7}$ into equal groups of $\frac{1}{2}$. Being able to draw or visualize where $\frac{1}{7}$ and $\frac{1}{2}$ fall on the number line is helpful in estimating the answer, but not in arriving at the precise answer, $\frac{2}{7}$.) Any understanding of fraction division, therefore, has to start from a more abstract level. With this in mind, we express the separation of 24 objects into 4 groups of 6s symbolically as 24 = 6 + 6 + 6 + 6, which is, of course, equal to 4 × 6, by the very definition of whole-number multiplication. Thus, the *division* statement 24/6 = 4 implies the *multiplication* statement 24 = 4 × 6.

At this point, we must investigate whether the multiplication statement 24 = 4 × 6 captures *all* of the information in the division statement 24/6 = 4. It does, because if we know 24 = 4 × 6, then we know 24 = 6 + 6 + 6 + 6, and therefore 24 can be separated into 4 groups of 6s. By the measurement meaning of division, this says 24/6 = 4. Consequently, the multiplication statement 24 = 4 × 6 carries exactly the same information as the division statement 24/6 = 4. Put another way, the *meaning* of 24/6 = **4** is 24 = **4** × 6. This is the symbolic reformulation of the concept of division of whole numbers that we seek.

This meaning of division is actually very clear from the standard algorithm for long division, as shown in the following example.

$$\begin{array}{r} 4 \\ 6\,\overline{)\,24} \\ \underline{-24} \\ 0 \end{array}$$

What we tell children is that to divide 24 by 6, we look for the number which, when multiplied by 6, gives 24. (Of course, children who have memorized the multiplication table of single-digit numbers will do this easily; those who haven't will struggle.)

In a similar fashion, the meaning of 36/12 = **3** is that 36 = **3** × 12, and the meaning of 252/9 = **28** is that 252 = **28** × 9, etc.

There is a subtle point here that is usually slurred over in the upper elementary grades but should be pointed out: in our examples, the dividend (be it 24, 36, or 252) is a multiple of the divisor, since otherwise the quotient cannot be a whole number. That said, now we can use abstract symbols† to express this new understanding of the division of whole numbers as follows: for whole numbers m and n, where m is a multiple of n and n is nonzero, the meaning of the division m/n = **q** is that m = **q** × n.

Beginning in fifth grade, we should teach students to reconceptualize division from this point of view. Their math teachers should help them revisit division from the perspective of this new knowledge and reshape their thinking accordingly. Such is the normal progression of learning.

Note that this reconceptualization is not a rejection of students' understanding of the division of whole numbers in their earlier grades. On the contrary, it evolves from that understanding and makes it more precise. This reconceptualization is important because the meaning of division, when reformulated this way, turns out to be universal in mathematics, in the following sense: if m and n are *any* two numbers (i.e., not just whole numbers) and n is nonzero, then the definition of "m divided by n equals q" is that m = q × n. In other words, m/n = **q** means m = **q** × n.

We now turn to fractions, a main source of math phobia. In the early grades, grades 2–4 more or less, students mainly acquire the vocabulary of fractions and use it for descriptive purposes (e.g., $\frac{1}{4}$ of a pie). It is only in grades 5 and up that serious learning of the *mathematics* of fractions takes place—and that's when students' fear of fractions sets in.

From a curricular perspective, this fear can be traced to at least two sources. The first is the loss of a natural reference point when students work with fractions. In learning to deal with the mathematics of whole numbers in grades 1–4, children always have a natural reference point: their fingers. But for fractions, the curricular decision in the United States has been to use a pizza or a pie as the reference point. Unfortunately, while pies may be useful in the lower grades, they are an awkward model for fractions bigger than 1 or for any arithmetic operations with fractions. For example, how do you multiply two pieces of pie or use a pie to solve speed or ratio problems?

A second source of the fear of fractions is the inherently abstract

*An example of partitive division is to put 24 items in 6 bags (each with an equal number of items), and find that each bag has 4 items.

†This definitely would be appropriate for fifth-graders once the idea of using symbols for abbreviations is introduced and *many* examples are given for illustration.

nature of the concept of a fraction. Whereas students' intuition of whole numbers can be grounded in counting their fingers, learning fractions requires a mental substitute for their fingers. By its very nature, this mental substitute has to be abstract because most fractions (e.g., $^{19}/_{13}$ or $^{251}/_{604}$) tend not to show up in the real world.

Because fractions are students' first serious excursion into abstraction,* understanding fractions is the most critical step in understanding rational numbers† and in preparing for algebra. *In order to learn fractions, students need to know what a fraction is.* Typically, our present math education lets them down at this critical juncture. All too often, instead of providing guidance for students' first steps in the realm of abstraction, we try in every conceivable way to ignore this need and pretend that there is no

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abstraction. When asked, *what is a fraction?*, we say it is just something concrete, like a slice of pizza. And when this doesn't work, we continue to skirt the question by offering more metaphors and more analogies: What about a fraction as "part of a whole"? As another way to write division problems? As an "expression" of the form m/n for whole numbers m and n (n > 0)? As another way to write ratios? These analogies and metaphors simply don't cut it. Fractions have to be numbers because we will add, subtract, multiply, and divide them.

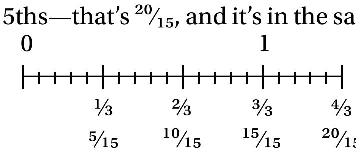
What does work well for showing students what fractions really are? The number line. In the same way that fingers serve as a natural reference point for whole numbers, the number line serves as a natural reference point for fractions.‡ The use of the number line has the immediate advantage of conferring coherence on the study of numbers in school mathematics: a number is now defined unambiguously to be a point on the number line.§ In particular, regardless of whether a number is a whole number, a fraction, a rational number, or an irrational number, it takes up its natural place on this line. (For the definition of fractions, including how to find them on the number line, see the sidebar on page 12.)

Now, let's describe the collection of numbers called fractions. Divide a line segment from 0 to 1 into, let's say, 3 segments of equal length; do the same to all the segments between any two consecutive whole numbers. These division points together with the whole numbers then form a sequence of equal-spaced points. These are the fractions with denominators equal to 3: the first division point to the right of 0 is what is called $^1/_3$, and the succeeding points of the sequence are then $^2/_3$, $^3/_3$, $^4/_3$, etc. The same is true for $^1/_n$, $^2/_n$, $^3/_n$, etc., for any nonzero whole number n. Thus, whole numbers clearly fall within the collection of numbers called fractions. If we reflect the fractions to the left of 0 on the number line, the mirror image of the fraction $^m/_n$ is by definition the negative fraction $-\,^m/_n$. Therefore, positive and negative fractions are now just points on the number line. Most students would find marking off a point $^1/_2$ of a unit to the left of 0 to be much less confusing than contemplating a negative $^1/_2$ piece of pie.

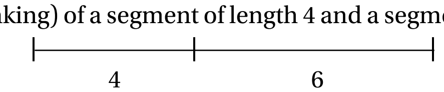
The number line is especially helpful in teaching students about the theorem on equivalent fractions, the single most important fact in the subject. To state it formally, for all whole numbers k, m, and n (where $k \neq 0$ and $n \neq 0$), $^m/_n = {}^{km}/_{kn}$. In other words, $^m/_n$ and $^{km}/_{kn}$ represent the same point on the number line. Let us consider an example to get a better idea: suppose m = 4, n = 3, and k = 5. Then the theorem asserts that

$$\frac{4}{3} = \frac{5 \times 4}{5 \times 3} \qquad \text{and, of course,} \qquad \frac{5 \times 4}{5 \times 3} = \frac{20}{15}.$$

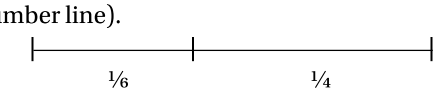
The number line makes the equality clear. To see how $^4/_3$ equals $^{20}/_{15}$, draw a number line and divide the space between 0 and 1, as well as between 1 and 2, into three equal parts. Count up to the 4th point on the sequence of thirds—that's $^4/_3$. Then take each of the thirds and divide them into 5 equal parts (an easy way to make 15ths). Count up until you get to the 20th point on the sequence of 15ths—that's $^{20}/_{15}$, and it's in the same spot as $^4/_3$.



The use of the number line has another advantage. Having whole numbers displayed as part of fractions allows us to see more clearly that the arithmetic of fractions is entirely analogous to the arithmetic of whole numbers. For example, in terms of the number line, 4 + 6 is just the total length of the concatenation (i.e., linking) of a segment of length 4 and a segment of length 6.



Then in the same way, we *define* $^1/_6 + {}^1/_4$ to be the total length of the concatenation of a segment of length $^1/_6$ and a segment of length $^1/_4$ (not shown in proportion with respect to the preceding number line).



We arrive at $^1/_6 + {}^1/_4 = {}^{10}/_{24}$ as we would if we were adding whole numbers, as follows. Using the theorem on equivalent fractions, we can express $^1/_6$ and $^1/_4$ as fractions with the same denominator: $^1/_6 = {}^4/_{24}$ and $^1/_4 = {}^6/_{24}$. The segment of length $^1/_6$ is therefore the concatenation of 4 segments each of length $^1/_{24}$, and the segment of length $^1/_4$ is the concatenation of 6 segments each of length $^1/_{24}$. The preceding concatenated segment is therefore the concatenation of (4 + 6) segments each of length $^1/_{24}$, i.e., $^{10}/_{24}$.** In this way,

(Continued on page 10)

*Very large numbers are already an abstraction to children, but children tend not to be *systematically* exposed to such numbers the way they are to fractions.

†Rational numbers consist of fractions and negative fractions, which of course include whole numbers.

‡See, for example, page 4-40 of the National Mathematics Advisory Panel's "Report of the Task Group on Learning Processes," www.ed.gov/about/bdscomm/list/mathpanel/report/learning-processes.pdf.

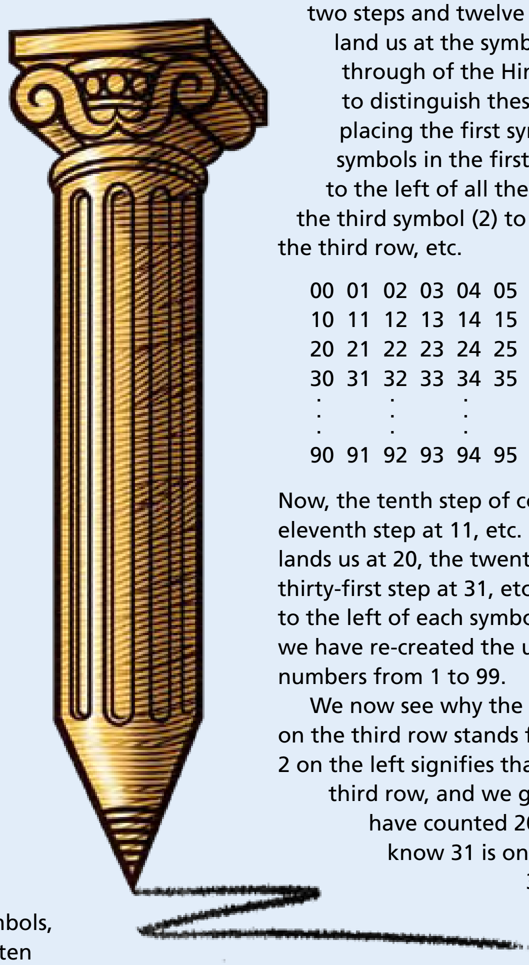
§We exclude complex numbers from this discussion, as they are not appropriate for the elementary grades.

**Naturally, the theorem on equivalent fractions implies that 10/24 = 5/12, as 10/24 = (2 × 5)/(2 × 12), but contrary to common belief, *the simplification is of no great importance*. Notice in particular that there was never any mention of the "least common denominator."

Understanding Place Value

Many teachers, rightly in my opinion, believe place value is the foundation of elementary mathematics. It is often taught well, using manipulatives such as base-10 blocks to help children grasp that, for example, the 4 in 45 is actually 40 and the 3 in 345 is actually 300.

But despite the importance of place value, the rationale behind it usually is not taught in colleges of education or in math professional development. That's probably because the deeper explanation is not appropriate for most students in the first and second grades, which is when place value is emphasized. But it is appropriate for upper-elementary students who are exploring number systems that are not base 10 (which often is done, without enough explanation, through games)—and it is certainly something that math teachers should know. So here it is: the sophisticated side of the simple idea of place value.



Let's begin with a look at the basis of our so-called Hindu-Arabic numeral system.* The most basic function of a numeral system is the ability to count to any number, no matter how large. One way to achieve this goal is simply to make up symbols to stand for larger and larger numbers as we go along. Unfortunately, such a system requires memorizing too many symbols, and makes devising a simple method of computation impossible. The overriding feature of the Hindu-Arabic numeral system, which will be our exclusive concern from now on, is the fact that it limits itself to using exactly ten symbols—0, 1, 2, 3, 4, 5, 6, 7, 8, 9—to do all the counting.† Let us see, for example, how "counting nine times" is represented by 9. Starting with 0, we go nine steps and land at 9, as shown below.

$$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9$$

But, if we want to count one more time beyond the ninth (i.e., ten times), we would need another symbol. Since we are restricted to the use of only these ten symbols, someone long ago got the idea of placing these same ten symbols next to each other to create more symbols.

The most obvious way to continue the counting is, of course, to simply recycle the same ten symbols over and over again, placing them in successive rows, as follows.

0 1 2 3 4 5 6 7 8 9
0 1 2 3 4 5 6 7 8 9
0 1 2 3 4 5 6 7 8 9
:   :   :   :   :

In this scheme, counting nine times lands us at the 9 of the first row, and counting one more time would land us at the 0 of the second row. If we want to continue counting, then the next step lands us at the 1 of the second row, and then the 2 of the second row, and so on.

However, this way of counting obviously suffers from the defect of ambiguity: there is no way to differentiate the first row from the second row so that, for example, going both two steps and twelve steps from the first 0 will land us at the symbol 2. The central breakthrough of the Hindu-Arabic numeral system is to distinguish these rows from each other by placing the first symbol (0) to the left of all the symbols in the first row, the second symbol (1) to the left of all the symbols in the second row, the third symbol (2) to the left of all the symbols in the third row, etc.

00 01 02 03 04 05 06 07 08 09
10 11 12 13 14 15 16 17 18 19
20 21 22 23 24 25 26 27 28 29
30 31 32 33 34 35 36 37 38 39
:   :   :   :   :
90 91 92 93 94 95 96 97 98 99

Now, the tenth step of counting lands us at 10, the eleventh step at 11, etc. Likewise, the twentieth step lands us at 20, the twenty-sixth step at 26, the thirty-first step at 31, etc. By tradition, we omit the 0s to the left of each symbol in the first row. That done, we have re-created the usual ninety-nine counting numbers from 1 to 99.

We now see why the 2 to the left of the symbols on the third row stands for 20 and not 2, because the 2 on the left signifies that these are numbers on the third row, and we get to them only after we have counted 20 steps from 0. Similarly, we know 31 is on the fourth row because the 3 on the left carries this information; after counting thirty steps from 0 we land at 30, and one more step lands us at 31. So the 3 of 31 signifies 30, and the 1 signifies one more step beyond 30.

With a trifle more effort, we can carry on the same discussion to three-digit numbers (or more). The moral of the story is that *place value is the natural consequence of the way counting is done in the decimal numeral system*.

For a fuller discussion, including numbers in arbitrary base, see pages 7–9 of *The Mathematics K–12 Teachers Need to Know* on my Web site at **http://math.berkeley.edu/~wu/School mathematics1.pdf**.

–H.W.

*This term is historically correct in the sense that the Hindu-Arabic numeral system was transmitted to the West from the Islamic Empire around the 12th century, and the Arabs themselves got it from the Hindus around the 8th century. However, recent research suggests a strong possibility that the Hindus, in turn, got it from the Chinese, who have had a decimal place-value system since time immemorial. See Lay Yong Lam and Tian Se Ang, *Fleeting Footsteps: Tracing the Conception of Arithmetic and Algebra in Ancient China* (Hackensack, NJ: World Scientific, 1992).

†Historically, 0 was not among the symbols used. The emergence of 0 (around the 9th century and beyond) is too complicated to recount here.

(Continued from page 8)

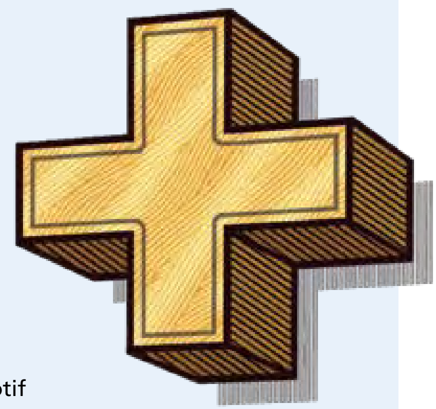
students get to see that fractions are the natural extension of whole numbers and not some confusing new thing. This realization smoothes the transition from computing with whole numbers to computing with fractions.

Hopefully this discussion has smoothed the transition for you too, because it's time for us to skip ahead to sixth grade and tackle division with fractions. Having learned to add, subtract, and multiply with fractions, students should be comfortable with fractions as numbers (just like whole numbers). So, their learning to divide with fractions can make use of the same scaffolding as learning to divide with whole numbers; students proceed from the simple to the complex. For example, a simple problem like $\frac{1}{2} \div \frac{1}{4} = 2$ could be taught using the measurement definition of division and showing students on the number line that $\frac{1}{4}$ appears twice in $\frac{1}{2}$. That's fine as an introduction, but ultimately, in order to prepare for more advanced mathematics, students must grasp a more abstract—and precise—definition of division with fractions. They must be able to answer the following question:

$$\text{Why does } \frac{5}{6} \div \frac{9}{4} \text{ equal } \frac{5}{6} \times \frac{4}{9}\text{ ?}$$

In other words, why invert and multiply? To give an explanation,

Teaching the Standard Algorithms



In the context of school mathematics, an algorithm is a finite sequence of explicitly defined, step-by-step computational procedures that end in a clearly defined outcome. The so-called standard algorithms for the four arithmetic operations with whole numbers are perhaps the best known algorithms.

At the outset, we should make clear that there is no such thing as *the unique* standard algorithm for any of the four operations +, −, ×, or ÷, because minor variations have been incorporated into the algorithms by various countries and ethnic groups. Such variations notwithstanding, the algorithms provide shortcuts to what would otherwise be labor-intensive computations, while the underlying mathematical ideas always remain the same. Therefore, from a mathematical perspective, the label "standard algorithms" is justified.

While it is easy to see why these algorithms were of interest before calculators became widespread, a natural question now is why we should bother to teach them. There are at least two reasons. First, without a firm grasp of place value and of the logical underpinnings of the algorithms, it would be impossible to detect mistakes caused by pushing the wrong buttons on a calculator. A more important reason is that, *in mathematics, learning is not complete until we know both the facts and their underlying reasons*. For the case at hand, learning the explanations for these algorithms is a very compelling way to acquire many of the basic skills as well as the abstract reasoning that are integral to mathematics. Both these skills and the capacity for abstract reasoning are absolutely essential for understanding fractions, decimals, and, therefore, algebra in middle school. One can flatly state that *if students do not feel comfortable with the mathematical reasoning used to justify the standard algorithms for whole numbers, then their chances of success in algebra are exceedingly small*.

These algorithms also highlight one of the basic tools used by research mathematicians and scientists: namely, that whenever possible, one should break down a complicated task into simple subtasks. To be specific, the leitmotif of the standard algorithms is as follows: *to perform a computation with multidigit numbers, break it down into several steps so that each step (when suitably interpreted) is a computation involving only single-digit numbers*. Therefore, a virtue of the standard algorithms is that, when properly executed, they allow students to ignore the actual numbers being computed, no matter how large, and concentrate instead on single digits.

This is an excellent example of the kind of abstract thinking that is critical to success in mathematics learning.

Building on the discussion of the addition algorithm given in the main article, we can further illustrate this leitmotif with the multiplication algorithm. In this case, let us assume that students already know the meaning of multiplication as repeated addition. The next step toward understanding multiplication requires that they know the multiplication table by heart—i.e., that they know the multiplication of single-digit numbers to automaticity. We now show, precisely, how this knowledge allows them to compute the product 257 × 48. First, observe that 257 = (2 × 100) + (5 × 10) + 7, so that by the distributive law [i.e., a(b + c) = ab + ac]:

$$257 \times 4 = (2 \times 4) \times 100 + (5 \times 4) \times 10 + (7 \times 4) \text{ and}$$
$$257 \times 8 = (2 \times 8) \times 100 + (5 \times 8) \times 10 + (7 \times 8).$$

Since they already know the single-digit products (2 × 4), (5 × 4), (7 × 4), (2 × 8), (5 × 8), and (7 × 8), and they know how to add, they can compute 257 × 4 and 257 × 8. Such being the case, we further note that 48 = (4 × 10) + 8, so that again by the distributive law:

$$257 \times 48 = (257 \times 4) \times 10 + (257 \times 8).$$

The right side being something they already know how to compute, they have therefore succeeded in computing 257 × 48 starting with a knowledge of the multiplication table. (For lack of space, we omit the actual writing out of the multiplication algorithm.)

Although the case of the long-division algorithm is more sophisticated, the basic principle is the same: it is just a sequence of single-digit computations.

For further details on the standard algorithms, see pages 38–90 of the first chapter of a professional development text for teachers that I am currently writing, available at **http://math.berkeley.edu/~wu/EMI1c.pdf**.

–H.W.

we have to ask what it means to divide fractions in the first place. The fact that *if we do not specify the meaning of dividing fractions, then we cannot possibly get a formula for it* should be totally obvious, yet this fact is not common knowledge in mathematics education. For such a definition, let us go back to the concept of division for whole numbers. Recall that in the case of whole numbers, having a clearly understood meaning for multiplication (as repeated addition) and division (as measurement division) allowed us to conclude that the meaning of the division statement $m/n = q$ for whole numbers m, n, and q ($n > 0$) is inherent in the multiplication statement $m = q \times n$. But now we are dealing with fractions, and the situation is different. To keep this article from becoming too long, let's assume that we already know how to multiply fractions,* but we are still searching for the meaning of fraction division. *Knowing that fractions and whole numbers are on the same footing as numbers*, it would be a reasonable working hypothesis that if $m/n = q$ means $m = q \times n$ for whole numbers m, n, and q, then the direct counterpart of this assertion in fractions should continue to hold. Now, if M, N, and Q are fractions ($N > 0$), we do not as yet know what $M/N = Q$ means, although we know the meaning of $M = Q \times N$ because we know how to multiply fractions. Therefore, the only way to make this "direct counterpart" in fractions come true is to use it as a *definition* of fraction division. In other words, we adopt the following definition: for fractions M and N ($N > 0$), the *division* of M by N, written M/N, is the fraction Q, so that $M = Q \times N$.

We'll get acquainted with this definition by looking at a special case. Suppose

$$\frac{5}{6} \div \frac{9}{4} = Q \text{ for a fraction } Q.$$

What could Q be? By definition, this Q must satisfy $\frac{5}{6} = Q \times \frac{9}{4}$. Now, recalling that $\frac{m}{n} = \frac{km}{kn}$ (the theorem on equivalent fractions), we use this fact to find Q by multiplying both sides of $\frac{5}{6} = Q \times \frac{9}{4}$ by $\frac{4}{9}$.

$$\begin{aligned} \frac{5}{6} \times \left(\frac{4}{9}\right) &= Q \times \frac{9}{4} \times \left(\frac{4}{9}\right) \\ &= Q \times \frac{(9 \times 4)}{(4 \times 9)} \\ &= Q \times 1 = Q \end{aligned}$$

This is the same as $\frac{5}{6} \times \frac{4}{9} = Q$. We can easily check that, indeed, this Q satisfies $\frac{5}{6} = Q \times \frac{9}{4}$. So, we see that

$$\frac{5}{6} \div \frac{9}{4} = \frac{5}{6} \times \frac{4}{9}$$

and we have verified the invert-and-multiply rule in this special case. But the reasoning is perfectly general, and it verifies in exactly the same way that for a nonzero fraction c/d, if $(a/b)/(c/d)$ is equal to a fraction Q, then Q is equal to $(a/b) \times (d/c)$. Therefore, the invert-and-multiply rule is always correct.

We have been staring at the concept of the division of fractions for quite a while, and we seem to be getting there because we have explained the invert-and-multiply rule. Therefore, it may be a little deflating to say that although we are getting *very* close, *we are not quite there yet*. There is a subtle point about the definition of fraction division that is still unsettled. This is something one should probably not bring up in a sixth-grade classroom, but which is, nevertheless, something a math teacher should be aware of. The question is whether, for arbitrary fractions M and N ($N > 0$), we can always divide M by N—i.e., whether there is always a *fraction* Q so that $M = Q \times N$. The answer, of course, is yes: if $M = a/b$ and $N = c/d$, then $Q = (a/b) \times (d/c)$ would do. So the upshot of all this is that we can *always* divide a fraction M by a nonzero fraction N, and the quotient, to be denoted by M/N, is the fraction obtained by the invert-and-multiply rule.

Once we know the meaning of division, we see there is nothing to the procedure of invert and multiply. What is sobering is that the rhyme, "Ours is not to reason why; just invert and multiply," gets it all wrong. With a precise, well-reasoned definition, there is no need to wonder why—the answer is clear. Thus, we return to our earlier theme: before we do anything in mathematics, we must make clear what it is we are doing. In other words, we must have a precise definition of division before we can talk about its properties. (And we must have a precise definition of fractions before we can expect students to do anything with them.)

The rhyme, "Ours is not to reason why; just invert and multiply," gets it all wrong. With a precise, well-reasoned definition, there is no need to wonder why—the answer is clear.

But one question remains: if division is just multiplication in a different format, why do we need division at all? The correct answer is that certain situations in life require it. An example of such a problem is the following:

> A 5-yard ribbon is cut into pieces that are each $\frac{3}{4}$ yard long to make bows. How many bows can be made?

Students usually recognize by rote that this problem calls for a division of 5 by $\frac{3}{4}$, but not the reason why division should be used. To better understand the reason for dividing, suppose the problem reads, instead, "A 30-yard ribbon is cut into pieces that are each 5 yards long. How many pieces can be made?" It would follow from the measurement interpretation of the division of whole numbers that the answer is $30/5 = 6$ pieces—i.e., there are six 5s in 30. The use of division for this purpose is well understood.

However, we are now dealing with pieces whose common length is a fraction $\frac{3}{4}$, and the reason for solving the problem by dividing 5 by $\frac{3}{4}$ is more problematic for many students. But if we use the preceding definition of division, the reason emerges with clarity. Suppose Q bows can be made from the ribbon. Here Q could be a fraction, and the meaning of "Q bows" can be explained by using an explicit example. If $Q = 6\frac{2}{3}$, for example, then "$6\frac{2}{3}$ bows" means 6 pieces that are each $\frac{3}{4}$ yard long, plus a piece that is the length of 2 parts when the $\frac{3}{4}$ yard is divided into 3 parts of equal length. *If multiplication is taught correctly*, so that the multiplication of two fractions is defined clearly, one can then explain why Q bows, no matter what fraction Q is, have a total length of $Q \times \frac{3}{4}$ yards. Therefore, if Q bows can be made from 5 yards of rib-

*The treatment of fraction multiplication in textbooks and in the education literature is mostly defective, but one can consult pages 62–74 of http://math.berkeley.edu/~wu/EMI2a.pdf for an introduction.

bon, then $5 = Q \times \frac{3}{4}$. *By the definition of fraction division*, this is exactly the statement that

$$\frac{5}{3/4} = Q.$$

This is the reason why division should still be used to solve this problem. Incidentally, the invert-and-multiply rule immediately leads to $Q = \frac{20}{3}$, which equals $6\,\frac{2}{3}$ pieces. In greater detail, that's 6 pieces and a leftover piece that is the length of 2 parts when $\frac{3}{4}$ yard is divided into 3 equal parts.

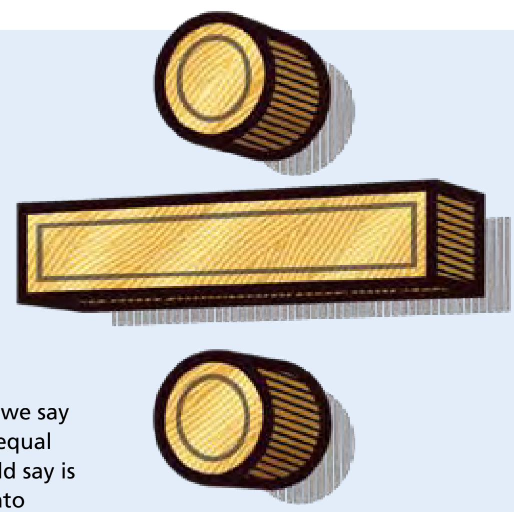
The Bigger Picture

At this point, I hope you can see that there's more to teaching elementary mathematics than is initially apparent. The fact is, there's much more to it than could possibly be covered in an article. But allow me to give you a glimpse of the bigger picture—of what elementary mathematics is really all about. I'll conclude with some of the latest thinking on the subject, thinking that points to mathematics teachers in the upper elementary grades being our best hope for providing all students the sound mathematics foundation they need.

Mathematics in elementary school is the foundation of all of K–12 mathematics and beyond. Therefore, to prepare students for all that is to come, it must, in a grade-appropriate manner, respect the basic characteristics of mathematics. What does this mean? To answer this question, we have to remember that the school mathematics curriculum, beginning with approximately grade 5, becomes increasingly engaged in abstraction and generality. It will no longer be about how to deal with a finite collection of numbers (such as, $\frac{1}{2} \times (27-11) + 56 = ?$), but rather about what to do with an *infinite* collection of numbers all at once (such as, is it true that $x^4 + y^4 = (x^2 + y^2 + \sqrt{2}xy)(x^2 + y^2 - \sqrt{2}xy)$ for *all* numbers x and y?). The progression of the topics, from fractions to negative fractions, and on to algebra, Euclidean geometry, trigonometry, and precalculus, gives a good indication that to learn mathematics, a student gradually must learn to cope with abstract concepts and precise reasoning, and must acquire a coherent overview of topics that are, cognitively, increasingly complex and diverse. For this reason, students in the upper elementary grades must be prepared for the tasks ahead by being slowly acclimatized to *coherence*, *precision*, and *reasoning*, although always in a way that is grade appropriate. Allow me to amplify each of these characteristics below.

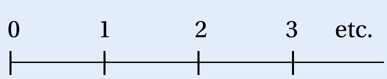
Coherence: If you dig beneath the surface, you will find that the

Defining Fractions

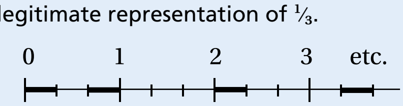


The precise definition of a fraction as a point on the number line is a refinement of, not a radical deviation from, the usual concept of a fraction as a "part of a whole." As I will explain, this refinement produces increased simplicity, flexibility, and precision.

Let us begin with a line, which is usually taken to be a horizontal one, and fix two points on it. The one on the left will be denoted by 0, and the one on the right by 1. (Because we will not take up negative numbers, our discussion will focus entirely on the half-line to the right of 0.) Now as we move from 0 to the right, we mark off successive points, each of which is as far apart from its neighbors as 1 is from 0 (like a ruler). Label these points by the whole numbers 0, 1, 2, 3, etc.



We begin with an informal discussion. If we adopt the usual approach to fractions, the "whole" would be taken to be the segment from 0 to 1, called the unit segment, to be denoted by [0, 1]. The number 1 is called the unit. Then a fraction such as $\frac{1}{3}$ would be, by common consent, 1 part when the whole [0, 1] is divided into 3 equal parts. So far so good. But if we try to press forward with mathematics, we immediately run into trouble because a fraction is a number—not a shape or a geometric figure. The unit segment [0, 1] therefore cannot be the whole. The language of "equal parts" is also problematic because in anything other than line segments, it usually is not clear what "equal parts" means. For example, if the whole is a ham, does "equal parts" mean parts with equal weights, equal lengths, equal amounts of meat, equal amounts of bones, etc.? So, we are forced to introduce more precision into our discussion in order to avoid misunderstanding. What we should specify, instead, is that the whole is the *length* of the unit segment [0, 1], rather than the segment itself. When we say [0, 1] is divided into "equal parts," what we should say is that [0, 1] is divided into segments of equal length. The fraction $\frac{1}{3}$ therefore would be the length of *any* segment so that three segments of the same length, when pieced together, form a segment of length 1. Since all segments between consecutive whole numbers have length 1, when we likewise divide each of the segments between consecutive whole numbers into 3 segments of equal length, the length of each of these shorter segments is also $\frac{1}{3}$. In particular, each of the following thickened segments has length $\frac{1}{3}$ and is therefore a legitimate representation of $\frac{1}{3}$.

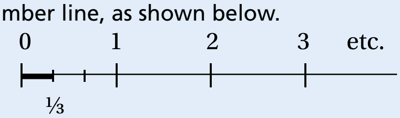


Now concentrate on the thickened segment on the far left. The distance of its right endpoint from 0 is naturally $\frac{1}{3}$. Since the value of each whole number on the number line reveals its distance from 0 (e.g., the distance of the point labeled 3 is exactly 3 from 0), logic demands that we label the right endpoint of this segment by the fraction $\frac{1}{3}$, and we call this main topics of the elementary curriculum are not a collection of unrelated facts; rather, they form a whole tapestry where each item exists as part of a larger design. Unfortunately, elementary school students do not always get to see such coherence. For example, although whole numbers and fractions are intimately related so that their arithmetic operations are essentially the same, too often whole numbers and fractions are taught as if they were unrelated topics. The comment I frequently hear that "fractions are such different numbers" is a good indication that elementary mathematics education, as it stands, cannot go forward without significant reform, such as the introduction of math teachers. Another example of the current incoherence is the fact that finite decimals are a special class of fractions, yet even in the upper elementary grades, decimals often are taught as a topic separate from fractions. As a result, students end up quite confused having to learn three different kinds of numbers (whole numbers, fractions, and decimals), whereas learning about fractions should automatically make them see that the other two are just more of the same. These are only two of many possible examples of our splintered curriculum and the great harm it does to students' learning.

Another manifestation of the coherence of mathematics is the ubiquity of the general principle of reducing a complicated task to a collection of simple subtasks. This principle runs right through all the standard algorithms, and also all the algorithms for decimals. In middle and high school mathematics, it also is the guiding principle in the discussion of congruence and similarity, provided these concepts are presented correctly. It also should be the guiding principle in the discussion of quadratic functions and their graphs, thereby making the basic technique of completing the square both enlightening and inevitable. Similarly, we saw how one embracing definition of division clarifies the meaning of the division of whole numbers and fractions and, as students should be taught in later grades, all rational numbers, real numbers, and complex numbers.

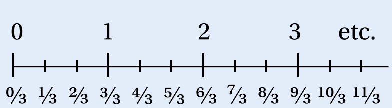
Precision: Children should learn about this mathematics tapestry in a language that does not leave room for misunderstanding or guesswork. It should be a language sufficiently precise so that they can reconstruct the tapestry step by step, if necessary. Too often, such precision of language is not achieved. For example, if you tell a sixth-grader that two objects are similar if they are the same shape but not necessarily the same size, it raises the question of what "same shape" means. A precise definition of similarity using

segment the "standard representation" of $\frac{1}{3}$. We also denote this thickened segment by $[0, \frac{1}{3}]$, because the notation clearly exhibits the left endpoint as 0 and the right endpoint as $\frac{1}{3}$. To summarize, we have described how the naive notion of $\frac{1}{3}$ as "1 part when the whole is divided into 3 equal parts" can be refined in successive stages and made into a point on the number line, as shown below.



In a formal mathematical setting, we now use this particular point as the official representative of $\frac{1}{3}$. In other words, whatever mathematical statement we wish to make about the fraction $\frac{1}{3}$, it should be done in terms of this point. This agreement enforces uniformity of language and lends clarity to any mathematical discussion about $\frac{1}{3}$. At the same time, the preceding discussion also gives us confidence that we can relate this point on the number line to our everyday experience with $\frac{1}{3}$, should that need arise.

What we have done to the representation of $\frac{1}{3}$ can be done to any fraction with a denominator equal to 3; for example, the standard representation of $\frac{2}{3}$ would be the marked point to the right of $\frac{1}{3}$ on the line above, and that of $\frac{3}{3}$ would be 1 itself. In general, *we identify any $\frac{m}{3}$ for any whole number m with its standard representation*, and we agree to let 0 be written as $\frac{0}{3}$. Here, then, are the first several fractions with denominators equal to 3.



Notice that it is easy to describe each of these fractions. For example, $\frac{7}{3}$ is the 7th division point when the number line is divided into thirds (in self-explanatory language). Equivalently, we can also say that $\frac{7}{3}$ is the 7th *multiple* of $\frac{1}{3}$ (again, in self-explanatory language).

What we have done to fractions with denominators equal to 3 can be done to any fraction. In this way, we transform the naive concept of a fraction as a part of a whole into the clearly defined concept of a fraction as a point on the number line. There are many advantages of this indispensable transformation, but there are three that should be brought out right away.

On the number line, all points are on equal footing, so that in the preceding picture, for example, there is no conceptual difference between $\frac{2}{3}$ and $\frac{11}{3}$ because both numbers are equally easy to access. The essence of this message is that, when a fraction is clearly defined as a point on the number line, the conceptual difference between so-called proper and improper fractions completely disappears. So the first major advantage of understanding fractions as points on the number line is that *all fractions are created equal*. Now we can discuss all fractions all at once with ease, whether proper or improper. In this small way, the concept of a fraction begins to simplify, and learning about fractions gets easier.

The second major advantage is that such a concept of fractions is inherently flexible. *Once we specify what the unit 1 stands for, all fractions can be interpreted in terms of the unit*. Now we are ready for that ham. If we let 1 stand for the weight of the ham, then $\frac{1}{3}$ would represent a piece of ham that is a third of the whole ham in weight. If, on the other hand, we let 1 stand for the volume of the ham, then the same fraction will now be a piece of ham that is a third of the whole ham in volume—e.g., in cubic inches.

This brings us to the third major advantage: *the increase in flexibility mandates an increase in precision*. Gone is the loose reference to "equal parts" in such a setting, because one must ask, equal parts in terms of what unit?

–H.W.

the concept of *dilation from a point A* would not allow for such confusion (as students will see that an object changes size, but not shape, when each point of the object is pushed away from or pulled into *A* by the same scaling factor).

Another example of the need for precision manifests itself in the way we present concepts. It is worth repeating that before we do anything in mathematics, we must make clear what it is that we are doing by providing precise definitions. There is no better example of the need for precision than the way fractions are generally taught in schools. Too often, fractions are taught without definitions, so that students are always in the dark about what fractions are. Thus, students multiply fractions without knowing what multiplication means and, of course, they invert and multiply, but dare not ask why. It is safe to hypothesize that such conceptual opaqueness is largely responsible for the notorious nonlearning of fractions—and, as a result, for great difficulty as students begin algebra.

> It is unrealistic to expect our generalist elementary teachers to possess this kind of mathematical knowledge—especially considering the advanced knowledge they must acquire to teach reading.

Reasoning: Above all, it is important that elementary school mathematics, like all mathematics, be built on reasoning. Reasoning is the power that enables us to move from one step to the next. When students are given this power, they gain confidence that mathematics is something they can do, because it is done according to some clearly stated, learnable, objective criteria. When students are emboldened to make moves on their own in mathematics, they become sequential thinkers, and sequential thinking drives problem solving. If one realizes that almost the whole of mathematics is problem solving, the centrality of reasoning in mathematics becomes all too apparent.

When reasoning is absent, mathematics becomes a black box, and fear and loathing set in. An example of this absence is some children's failure to shift successive rows one digit to the left when multiplying whole numbers, such as on the left below.

```
      826            826
  ×   473        ×   473
     2478           2478
     5782          5782
  +  3304       + 3304
    11564         390698
```

If no reason is ever given for the shift, it is natural that children would take matters into their own hands by making up new rules. Worse, such children miss an excellent opportunity to deepen their understanding of place value and see that, in this example, the multiplication 4 × 8 is actually 400 × 800, and that this is the basic reason underlying the shift. Another notorious example is the addition of fractions by just adding the numerators and the denominators, something that happens not infrequently even in college. Learning cannot take place in the classroom if students are kept in the dark about why they must do what they are told to do.

The characteristics of coherence, precision, and reasoning are not just niceties; they are a prerequisite to making school mathematics learnable. Too often, all three are absent from elementary curricula (at least as they are sketched out in both state standards and nationally marketed textbooks).[*] As a result, too often they also are absent from the elementary classroom. The fact that many elementary teachers lack the knowledge to teach mathematics with coherence, precision, and reasoning is a systemic problem with grave consequences. Let us note that this is not the fault of our elementary teachers. Indeed, it is altogether unrealistic to expect our generalist elementary teachers to possess this kind of mathematical knowledge—especially considering all the advanced knowledge of how to teach reading that such teachers must acquire. Compounding this problem, the pre-service professional development in mathematics is far from adequate.[†] There appears to be no hope of solving the problem of giving all children the mathematics education they need without breaking away from our traditional practice of having generalist elementary school teachers.

The need for elementary teachers to be mathematically proficient is emphasized in the recent report of the National Mathematics Advisory Panel.[‡] Given that there are over 2 million elementary teachers, the problem of raising the mathematical proficiency of all elementary teachers is so enormous as to be beyond comprehension. A viable alternative is to produce a much smaller corps of *mathematics teachers* with strong content knowledge who would be solely in charge of teaching mathematics at least beginning with grade 4. The National Mathematics Advisory Panel has taken up this issue. While the absence of research evidence about the effectiveness of such mathematics teachers precluded any recommendation from that body, the use of mathematics teachers in elementary school was suggested for exactly the same practical reasons.[§] Indeed, this is an idea that each state should seriously consider because, for the time being, there seems to be no other way of providing our children with a proper foundation for mathematics learning.

We have neglected far too long the teaching of mathematics in elementary school. The notion that "all you have to do is add, subtract, multiply, and divide" is hopelessly outdated. We owe it to our children to adequately prepare them for the technological society they live in, and we have to start doing that in elementary school. We must teach them mathematics the right way, and the only way to achieve this goal is to create a corps of teachers who have the requisite knowledge to get it done. ☐

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[*]See, for example, the National Mathematics Advisory Panel's "Report of the Task Group on Conceptual Knowledge and Skills," especially Appendix B, www.ed.gov/about/bdscomm/list/mathpanel/report/conceptual-knowledge.pdf, and "Report of the Subcommittee on Instructional Materials," www.ed.gov/about/bdscomm/list/mathpanel/report/instructional-materials.pdf.

[†]See, for example, the National Council on Teacher Quality's *No Common Denominator*, www.nctq.org/p/publications/reports.jsp.

[‡]See Recommendation 7 on page xviii and Recommendations 17 and 19 on page xxi in *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*, www.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf.

[§]*Foundations for Success*, Recommendation 20, page xxii, see note above for URL.