Ask the Cognitive Scientist

Do Manipulatives Help Students Learn?

How does the mind work—and especially how does it learn? Teachers' instructional decisions are based on a mix of theories learned in teacher education, trial and error, craft knowledge, and gut instinct. Such knowledge often serves us well, but is there anything sturdier to rely on?

Cognitive science is an interdisciplinary field of researchers from psychology, neuroscience, linguistics, philosophy, computer science, and anthropology who seek to understand the mind. In this regular American Educator *column, we consider findings from this field that are strong and clear enough to merit classroom application.*

By Daniel T. Willingham

Question: Is there any reason to be cautious when using manipulatives in class? I understand that some educators might have mistakenly thought that manipulatives—concrete objects that students handle mostly during math and science lessons—help because they give kinesthetic learners the hands-on experiences they need, and we now know that theory is wrong.[1] Still, isn't it the case that *all* small children learn better via concrete objects than via abstractions? Surely it helps students focus if classroom activities are mixed up a bit, rather than listening to endless teacher talk.

Daniel T. Willingham is a professor of cognitive psychology at the University of Virginia. He is the author of When Can You Trust the Experts? How to Tell Good Science from Bad in Education *and* Why Don't Students Like School? *His most recent book is* Raising Kids Who Read: What Parents and Teachers Can Do. *For his articles on education, go to www.danielwillingham.com. Readers can post questions to "Ask the Cognitive Scientist" by sending an e-mail to ae@aft.org. Future columns will try to address readers' questions.*

Answer: Research in the last few decades has complicated our view of manipulatives. Yes, they often help children understand complex ideas. But their effectiveness depends on the nature of the manipulative and how the teacher encourages its use. When these are not handled in the right way, manipulatives can actually make it harder for children to learn.

In 1992, in the pages of this magazine, Deborah Loewenberg Ball warned against putting too much faith in the efficacy of math manipulatives.* At the time, research on the topic was limited, but Ball noted the unwarranted confidence among many in the education world that "understanding comes through the fingertips." (Manipulatives might also make ideas more memorable; here, I'll focus on whether they aid the understanding of novel ideas.) Ball explained how the embodiment of

*See "Magical Hopes" in the Summer 1992 issue of *American Educator*, available at www.aft.org/ae/summer1992/ball.

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a mathematical principle in concrete objects might be much more obvious to adults who know the principle than to children who don't. We see place value, whereas they see bundles of popsicle sticks. And isn't the *lesson*, Ball asked, what really matters—not the manipulative, but how the teacher introduces it, guides its use, and shapes its interpretation?

Twenty-five years later, enthusiasm for manipulatives remains strong, especially in math and science.[2] For example, a joint statement from the National Association for the Education of Young Children and the National Council of Teachers of Mathematics advises, "To support effective teaching and learning, mathematics-rich classrooms require a wide array of materials for young children to explore and manipulate."[3] Teachers seem to heed this advice. Empirical data are scarce, but surveys of teachers indicate that they think it's important to use manipulatives, and early elementary teachers report using them nearly every day.[4]

While enthusiasm for manipulatives seems not to have changed since 1992, the research base has. It shows that, although manipulatives frequently help children understand concepts, they sometimes backfire and prompt confusion.[5] Instead of starting with a catalogue of instances in which manipulatives help (or don't), let's first consider the theories meant to explain how manipulatives influence children's thinking. Research has shown that two prominent theories are likely wrong. A third theory is more solid, and will provide a useful framework for us to consider some research findings. That, in turn, will provide guidance for classroom use of manipulatives.

Manipulatives often help children understand complex ideas.

Why Do Manipulatives Help?

Why might a child learn a concept when it is instantiated in physical materials that can be manipulated, whereas the same concept in symbolic form confounds the child? Jerome Bruner and, even more prominently, Jean Piaget offered answers rooted in the nature of child development.[6] They suggested that young children think more concretely than older children or adults. Children depend on physically interacting with the world to make sense of it, and their capability to think abstractly is absent or, at best, present only in a crude form. The concrete/abstract contrast forms one of the vital differences between two stages of cognitive development in Piaget's theory. In the concrete operational stage (from about age 7 to 12), the child uses concrete objects to support logical reasoning, whereas in the formal operations stage (age 12 to adulthood), the child can think using pure abstractions.

But much research in the last 50 years has shown that this characterization of children's thought is inaccurate. Consider children's understanding of numbers. Piaget suggested that preschoolers have no understanding of numbers as an abstraction—they may recite counting words, but they don't have the cognitive representation of what number names really refer to.[7]

But later work showed that although children may make mistakes in counting, the way they count shows abstract knowledge of what counting is for and how to do it. When counting, they assign one numeric tag to each item in a set, they use the same tags in the same order each time, they claim that the last tag used is the number of items in the set, and they apply these rules to varied sets of objects.[8] Preschoolers show abstract thinking in other domains as well, for example, their understanding of categories like "living things."[9] So it's not the case that children's thinking is tethered to concrete objects.

Another theory suggests that manipulatives help because they demand movement of the body. Some researchers propose that cognition is not a product of the mind alone, but that the body participates as well. In these theories, not all mental representations are completely abstract, but rather may be rooted in perception or action. For example, we might think that we have an abstract idea of what "blue" means, or what is meant when we hear or read the word "kick." But some evidence suggests that thinking of "blue" depends on the same mental representation you use when you actually perceive blue. The meaning of the word "kick" depends on what it feels like to actually kick something.[10]

By this account, manipulatives are effective because their demand for movement is in keeping with the way that thought is represented. If this theory is right, then instructional aids similar to manipulatives that aren't actually manipulated shouldn't help—it's the movement that really matters. The last decade has seen a great deal of research on that question; do computer-based, virtual manipulatives work as well as the real thing? Although there are exceptions,[11] computer-based manipulatives usually help students as much as physical ones.[12] These findings don't mean that movement is completely unrelated to cognition, but they make it doubtful that movement underpins the efficacy of manipulatives.

Furthermore, and crucial to our purposes, both theories—children are concrete thinkers, and physical movement is central to thought—seem to predict that manipulatives will always lead to better understanding. As we'll see, manipulatives are often helpful, but not always.[13]

A third theory provides a better fit to the data. It suggests that manipulatives help children understand and remember new concepts because they serve as analogies; the things manipulated are symbols for the new, to-be-understood idea. This hypothesis is a bit counterintuitive, because we think of manipulatives working exactly because they are easily understood, readily interpretable. But they are not to be interpreted literally. Popsicle sticks or counters or rods are *symbols* for something else.[14] A set of popsicle sticks reifies the concept of number, which is abstract and difficult for the young child to wrap his or her mind around. Manipulatives are used so often in math and science exactly because those subjects are rife with unintuitive concepts like number, place value, and velocity.[15]

But their effectiveness depends on the nature of the manipulative and how the teacher encourages its use.

Analogies help us understand difficult new ideas by drawing parallels to familiar ideas. For example, children are already familiar with fractions in some contexts. They may not have the words to describe their thinking, but they understand that a pizza can be considered a whole that is divisible by eight slices, and that when each of two people take four slices, they divide the pizza equally. The manipulative, then, calls on an existing memory (of pizza) and uses it as a metaphor, extending this existing knowledge to something new (the abstract idea of fractions).[16]

The data that posed a problem for other theories are no problem here: this theory doesn't predict that children can't think abstractly, and it doesn't accord any special role to moving the body. Indeed, this theory sits comfortably with other studies showing that embedding problems in familiar situations helps students, even if there is nothing to manipulate physically or virtually.

For example, one study compared how well novices solved algebra problems in symbolic form and when problems were embedded in a familiar scenario.[17] Some students saw "Solve for X, where X = .37(7) + .22," and others read "After buying donuts at Wholey Donuts, Laura multiplies the 7 donuts she bought by their price of $0.37 per donut. Then she adds the $0.22 charge for the box they came in and gets the total amount she paid. How much did she pay?" Students in the latter condition were more successful than those in the former.

In the next section we put this theory to work. Manipulatives sometimes flop when common sense would have us believe they ought to help. Thinking of manipulatives as analogies clarifies what might otherwise be a confusing pattern of experimental results.

Manipulatives Aid Understanding When Attention Is on the Relevant Feature

It seems obvious that children must attend to a manipulative if it is to work, and much research has focused on manipulatives' perceptual richness (i.e., whether they are colorful and visually complex) because perceptual richness can draw the student's attention. For example, in one study, researchers had fifth-graders solve mathematical word problems involving money.[18] Some students were given play money as manipulatives to use while working the problems; these would be considered perceptually rich because they were printed with lots of detail. Other children were also given coins and bills as manipulatives, but they were bland: simple slips of white paper with the monetary value written on them. A third group received no manipulatives. The researchers didn't just count the number of problems correctly worked; they also differentiated types of errors when students got a problem wrong: conceptual errors (where students set up the math incorrectly) or nonconceptual (e.g., copying the information inaccurately, adding two digits incorrectly, forgetting to show one's work). Researchers found students made fewer conceptual errors when using the perceptually rich materials. (They also made many *more* nonconceptual errors, a point to which we will return.)

Another experiment concerning attention and perceptual richness focused on 3- to 4-year-olds learning numerical concepts. Two sets of counters were placed on a table, and a crocodile was to be positioned so that it would "eat" the numerically larger set.[19] Researchers found that children learned more from the game if the counters were perceptually rich (realistic-looking frogs) instead of bland (simple green counters).

But in addition to varying the counter, experimenters also examined the role of instruction. In one condition, the experimenter acted as a player, taking turns with the child. In the other, the experimenter modeled how to play and provided feedback after the child's turn. In this second condition, the instruction guided attention effectively. With it, children using the bland counters learned as much as those using the perceptually rich counters. Again, the child's attention is thought to be critical; it can be drawn by the perceptually rich materials, or directed by the teacher.

In some instances, the guidance of attention may be less explicit by simply instructing the student how the manipulative is to be used, which in turn makes attention to the right feature of the manipulative likely. Consider use of a physical, numbered line to help understand the concept of addition. Given the problem 6 + 3, the child might find 6 and then count "1, 2, 3," and so find the answer, 9. But using the manipulative that way does not focus the child's attention on the continuity of numbers. A better method is to find 6, and then count "7, 8, 9."[20]

Researchers tested this idea by having kindergartners play a game similar to Chutes and Ladders, with a 10 by 10 array of numbers from 1 to 100 on a game board that players were to progress through, with a spinner determining the number of spaces to move on each turn.[21] They instructed some children to count out their moves from 1; that is, if they were on number 27 on the game board and spun a 3, they were to count aloud "1, 2, 3." Other children were asked to count from the initial number, i.e., "28, 29, 30." After two weeks of game play, the latter group showed significant gains in number understanding, compared with the former group.

Bruner thought teacher guidance was crucial for manipulatives to aid learning.[22] He suggested that students were unlikely to learn the target concepts if they were simply given the materials and encouraged to do with them what they wished. Bruner's caution is in keeping with other research on pure discovery learning. When children are given little guidance in the hope that they will, in the course of loosely structured exploration, discover key concepts in math and science, outcomes are usually disappointing, compared with situations using more explicit instruction.[23] At the same time, overly restrictive, moment-by-moment instructions about exactly what to do with manipulatives might be expected to backfire as well; this practice raises the risk that students would simply follow the teacher's directions without giving the process much thought.[24]

Manipulatives Don't Aid Understanding When Attention Is Not on the Relevant Feature

We might think that perceptually rich manipulatives are always the way to go. Why use green dots when you can use frogs? *Of course* frogs are going to be more engaging for students! But that conclusion would be hasty. Remember, manipulatives are analogies, and analogies are usually imperfect. In an analogy, an unfamiliar, to-be-learned idea (e.g., fractions) is likened to a familiar idea (e.g., pizza) because they share one or more important qualities (e.g., divisibility). But pizzas have lots of qualities that you would not want to impute to fractions: they are edible, they are purchasable, they are often found at parties, and so on. So it's not enough that a manipulative call attention to itself by being perceptually rich; it must call attention to the key feature, and not to other features. And indeed, manipulatives fail to aid understanding when children focus attention on a feature that is irrelevant to the analogy. There are several ways that might happen.

First, the manipulative might simply be poorly designed in that it's missing the crucial feature. A series of experiments has shown that playing a board game with numbers arrayed linearly helps children understand some properties of numbers.[25] The benefit is obvious because we recognize the game is analogous to the number line. But if the game board's numbers are arranged in a circle instead of a line, children don't benefit.[26]

Second, the manipulative might have the relevant feature, but the child does not attend to it because some other feature is more salient. This is where perceptual richness can backfire. Imagine Cuisenaire rods (meant to help children understand number concepts) painted to look like superhero action figures. Students could hardly be blamed if they failed to focus on the differing length of the rods, which is their important symbolic feature.*

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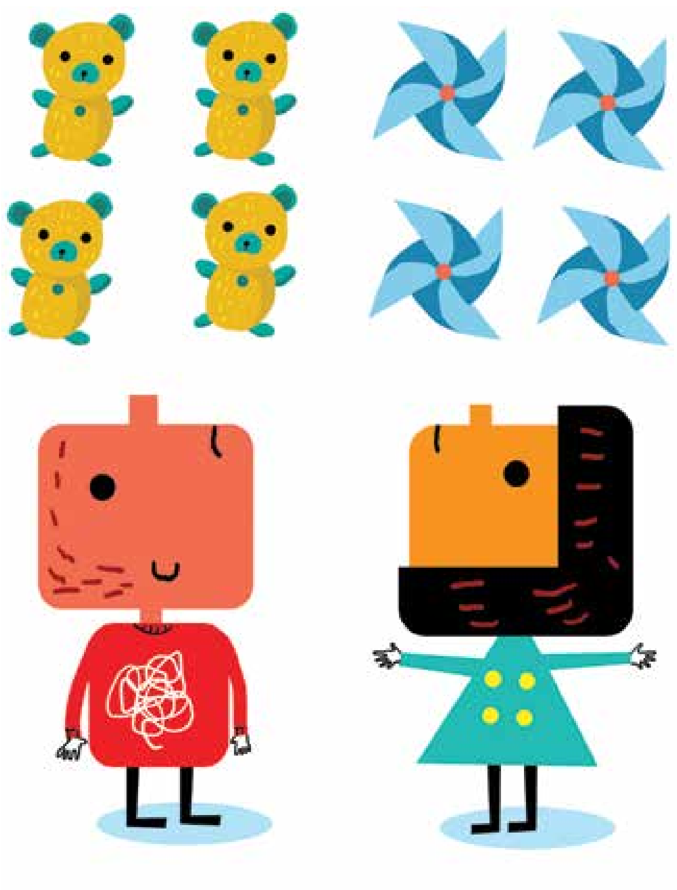
But the feature doesn't need to be that obviously distracting to confuse children. *The child has no way of knowing which features of the manipulative are important and which are not.* If the teacher uses apples as counters, is it important that apples are roughly spherical? That we know what the inside looks like, even though it's not visible?[27] Recall the experiment mentioned earlier using play money. Perceptually rich manipulatives reduced conceptual errors (children set up the math problem correctly) but *increased* other types of errors (e.g., calculation errors). Detailed manipulatives draw attention (which helps) but then may direct attention to irrelevant details (what Washington looks like on the bill).

Third, even if the child knows which feature of the manipulative is relevant, it may be difficult to keep in mind that it is a symbol. In the play money experiment, the children already had some experience with real money, and the play money was meant to serve the same purpose familiar to them. More often, the symbolic connection is new. A child is used to thinking of a slice of pie as something to eat. Now it's supposed to represent the abstract idea "⅛ of a whole."

Research has shown that this duality poses a problem. Researchers asked 3- and 4-year-olds to perform a counting task using manipulatives.[28] The manipulatives varied in their perceptual richness and in children's familiarity with the object: Some children were given objects to use as counters that were perceptually rich and familiar (e.g., small animal figurines). Others got objects that were familiar, but not perceptually rich (popsicle sticks). Still others got counters that were unfamiliar and perceptually rich (multicolored pinwheel blades) or counters that were unfamiliar and not

*For more on how embellishment can be distracting, see "Keep It Simple to Avoid Data Distractions" in the Summer 2013 issue of *American Educator*, available at www.aft.org/ae/summer2013/notebook.



perceptually rich (monochrome plastic chips).

The researchers observed a substantial disadvantage in the counting task for children using the animal figurines, compared with the other groups. As we've seen in previous experiments, richness drew attention to the manipulative, just as it did in the play money experiment. In that case, the children were meant to think of the manipulative (play money) in the same way they thought of its symbolic referent (real money). But children already know animal figurines to be toys, which one plays with. It's hard to also think of them as counters representing the abstract concept of number. The perceptually rich pinwheel blades did not pose the same problem because, even though they drew the child's attention, they were unfamiliar; it was easier to think of them as a symbol for something else, because the child did not think of them as having another purpose.

Thinking of an object as having two meanings overwhelms working memory in young children. This interpretation is supported by other landmark work on mental representation. In the standard paradigm, children are shown a diorama of a room and are told it is an exact model of a larger room that they will be shown. Then the experimenter hides a small Snoopy doll in the diorama and says that big Snoopy will be hiding in exactly the same place in the large room.[29] The child is then taken to the large room (which is, indeed, identical in every way to the diorama, except for size) and is encouraged to find large Snoopy. Two-and-a-half-year-olds are terrible at this task. But they improve dramatically if they are shown the diorama behind a pane of glass; that makes them less likely to think of the diorama as a toy, leaving the child free to see it as a symbol. And 3-year-olds (who normally perform pretty well on the task) are worse at finding big Snoopy if they are prompted to think of the diorama as a toy by encouraging them to play with it before searching for big Snoopy.[30]

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Moving Beyond the Manipulative

Obviously, our intention in using manipulatives is not to make children forever dependent on them; we don't expect a high school student to pull out strings of beads as he or she prepares to do math homework. It's not just that manipulatives are time-consuming and inconvenient to use. They also fail to apply to an entire domain. Helping a child understand the idea of fractions by dividing a circular pizza or pie works well until you encounter a fraction with the denominator 9. Or 10,000. Or suppose a teacher uses colored chips to model counting and addition: black chips represent positive numbers and red chips are negative numbers. This manipulative leads to intuitive representations for many problems, but not for all. How would you represent 5 + (−3)? Five black chips and three red chips?

These might seem like phantom problems. We use manipulatives because we believe they will aid student understanding. We expect using pizza manipulatives will give students the conceptual understanding of fractions that they will then transfer to the symbolic representation, so they won't need a manipulative for a fraction with a denominator of 10,000. We expect that the conceptual knowledge will successfully apply to other concrete representations, like calculating how many books can fit on a bookshelf. Alas, it's not so simple.

As we've seen, manipulatives that are perceptually rich draw attention to themselves, which can be good because they could highlight the right properties. For example, a "10s" rod is 10 times the length of a "1s" rod. In another example, college undergraduates were taught a principle of self-organization called competitive specialization, which is applicable to ant foraging. An interactive computer simulation depicted ants foraging for fruit, and students learned more quickly if the ants and fruit looked realistic (rather than being depicted as dots and color patches).[31]

But crucially, the study showed that transfer to a conceptually similar problem is *worse* with the realistic-looking ants than with the dots. Other work confirms that generalization. Undergraduates were taught a new math concept (commutative mathematical group of order 3) either using geometric shapes that were meaningless to the principle, or using sym-

bols (cups of water) about which students had prior knowledge that was applicable to learning the new concept. Sure enough, students learned the concept more quickly with the familiar symbols, but transfer to different problems was better with the abstract symbols.[32]

Even if students learn a concept with manipulatives and simultaneously learn it with written symbols, the two may remain separate, with students never drawing the connection between them. This duality would explain the results of a yearlong study of third-graders using Dienes blocks (and other manipulatives) in their math classroom.[33] The researchers found that most children became proficient in using the blocks to solve problems, but those who were most proficient were actually the worst in working the same problems with standard written notation. It was as if using the blocks stayed mentally separate from the symbolic representation.

Students are more likely to understand the concept the manipulative is meant to convey if that parallel is made explicit to them.

What guidance can this research review offer to classroom practice? A simple review of key conclusions makes a few things clear. First, we must temper our endorsement of manipulatives in classrooms with some caveats; there are instances where manipulatives will not speed children's learning, and may even slow it down. Second, the objects themselves should draw attention to whichever feature is meant to convey information, for example, the length of a rod if it is meant as an analogy to number. Third, teachers should provide instruction in the use of the manipulative so that this feature is salient to students, but teachers should not be so controlling that students are merely executing instructions without thinking. In addition, students are more likely to understand the concept the manipulative is meant to convey if that parallel is made explicit to them.

Two other ideas have less direct empirical support but are worth considering.

You'll recall there was a tradeoff between the perceptual richness of the object used as a manipulative and the likelihood of successful transfer of learning. Students were quicker to learn the foraging principle when illustrated with realistic-looking ants, but the knowledge then seemed stuck to the example with the ants.

A principle known as concreteness fading might address this problem. Originally proposed by Bruner,[34] the idea is that instruction begins with concrete, perceptually rich manipulatives, and students gradually move to more abstract symbols.[35] The Singapore math method offers an example.[36] Preschoolers might initially use stuffed animals when working with number concepts, then animal stickers, then plain circular stickers, and then square blocks appended to form a line. Although concreteness fading was proposed 50 years ago, empirical research confirming the utility of this intuitively appealing idea is limited.

Another idea that seems like it ought to work (and yet has limited experimental backing) is the consistent use of the same set of manipulatives for the same concept. It's tempting for a teacher to use stickers as counters one day, Cheerios another, and so on. It adds some variety and would, it would seem, boost student engagement.

But thinking of manipulatives as analogies suggests student comprehension will be better if there is consistency between manipulatives and what they are to represent. Concreteness fading might be used to get students to the point of thinking of black chips as number units, for example, and, thereafter, they are used anytime number units are invoked. That reduces the memory load for students, allowing them to benefit fully from their previous work. ☐

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