Inventing Numbers

How Mathematicians Filled the Inky Void

By David Berlinski

The most familiar of objects, numbers are nonetheless surprisingly slippery, their sheer slipperiness interesting evidence that certain intellectual tools may be successfully used before they are successfully understood. Numbers tend to sort themselves out by clans or systems, with each new system arising as the result of a perceived infirmity in the one that precedes it.

The natural numbers 1, 2, 3, 4,…, start briskly at 1 and then go on forever, although how we might explain what it means for anything to go on forever without in turn using the natural numbers is something of a mystery. In almost every respect, they are, those numbers, simply given to us, and they express a primitive and intimate part of our experience. Like so many gifts, they come covered with a cloud. Addition makes perfect sense within the natural numbers; so, too, multiplication. Any two natural numbers may be added, any two multiplied. But subtraction and division are curiously disabled operations. It is possible to subtract 5 from 10. The result is 5. What of 10 from 5? No answer is forthcoming from *within* the natural numbers. They *start* at 1.

The integers represent an expansion, a studied enlargement, of the system of natural numbers, one motivated by obvious intellectual distress and one made possible by two fantastic inventions. The distress, I have just described. And those inventions? The first is the number 0, the creation of some nameless but commanding Indian mathematician. When 5 is taken away from 5, the result is nothing whatsoever, the apples on the table vanishing from the table, leaving in their place a peculiar and somewhat perfumed absence. What *was* there? Five apples. What *is* there? Nothing, *Nada,* Zip. It required an act of profound intellectual audacity to assign a name and hence a symbol to all that nothingness. Nothing, *Nada,* Zip, Zero, 0.

The negative numbers are the second of the great inventions. These are numbers marked with a caul: –504, –323, –32, –1. The result is a system that is centered at 0 and that proceeds toward infinity in both directions:…, –5, –4, –3, –2, –1, 0, 1, 2, 3, 4,…. Subtraction is now enabled. The result of taking 10 from 5 is –5.

And yet if subtraction (along with addition and multiplication) is enabled among the integers, division still provokes a puzzle. Some divisions may be expressed entirely in integral terms—12 divided by 4, for example, which is simply 3. But what of 12 divided by 7? In terms of the integers, it is nothing whatsoever and so calls to mind those moments on *Star Trek* when the transporter fails and causes the Silurian ambassador to vanish.

It is thus that the rational numbers, or fractions, enter the scene, numbers with a familiar doubled form: $\frac{2}{3}$, $\frac{5}{9}$, $\frac{17}{32}$. The fractions express the relationship between the whole of things that have parts and the parts that those things have. There is that peach pie, the luscious whole, and there are those golden dripping slices, parts of the whole, and so two-thirds or five-ninths or seventeen-thirty-seconds of the thing itself. With fractions in place, division among the integers proceeds apace. Dividing 12 by 7 yields the exotic $\frac{12}{7}$, a number that does not exist (and could not survive) amidst the integers. But fractions play in addition a conspicuous role in measurement and so achieve a usefulness that goes beyond division.

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How Much and How Many

The natural numbers answer the oldest and most primitive of questions—*how many?* It is with the appearance of this question in human history that the world is subjected for the first time to a form of conceptual segregation. To count is to classify, and to classify is to notice and then separate, things falling within their boundaries and boundaries serving to keep one thing distinct from another.

The rational numbers, on the other hand, answer a more modern and sophisticated question—*how much?* Counting is an all or nothing affair. Either there are three dishes on the table, three sniffling patients in the waiting room, three aspects to the deity, or there are not. The question *how many?* does not admit of refinement. But *how much?* prompts a request for measurement, as in *how much does it weigh?* In measurement some extensive quantity is assessed by means of a scheme that may be made better and better, with even the impassive and uncomplaining bathroom scale admitting of refinement, pounds passing over to half pounds and half pounds to quarter pounds, the whole system capable of being *forever* refined were it not for the practical difficulty of reading through the hot haze of frustrated tears the awful news down there beneath all that blubber. This refinement, which is an essential part of measurement, plainly requires the rational numbers for its expression and not merely the integers. I may *count* the pounds to the nearest whole number; in order to *measure* the fat ever more precisely, I need those fractions.

With fractions in place, the system of numbers in which they are embedded undergoes a qualitative change. The integers are discrete in the sense that between 1 and 2 there is absolutely nothing. There is not much more, needless to say, between 2 and 3. Going from one integer to another is like proceeding from rock to rock across an inky void. The fractions fill up the spaces in the void, with $\frac{3}{2}$, for example, standing solidly between 1 and 2. There are now rocks between rocks—the void is vanishing—and rocks between rocks and rocks, with $\frac{1}{3}$ standing between $\frac{1}{4}$ and $\frac{1}{2}$. The filling-in of fractions between fractions is a process that goes on forever. That void has vanished. The number system is now *dense,* and not discrete, infinite in either direction (as the positive and negative integers go on and on) and infinite between the integers as well. In looking at the space between 1 and 2, swarming now with pullulating fractions, the mathematicians, or the reader, may for a moment have the unexpected sensation of peering into some sinister sinkhole—some hidden source of creation.

The Black Blossoms of Geometry: Inscribing Numbers on the Number Line

Geometry is a world within the world. The integers and the fractions represent the numbers with which that world must be coordinated. But geometry is one thing, arithmetic another. Taken on their own, they remain alien, one to the other. Analytic geometry represents a program in which arithmetic comes vibrantly to life within geometry, and so describes a process in which an otherwise severe world is made to blossom.

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The program of analytic geometry is to evoke the numbers from the stubby soil of a geometrical landscape; it begins with a solitary line, something that lies in the imagination like a straight desert highway stretching from one blue horizon to the other. The traveler drifting down that highway, it is worth remembering, requires only *one* landmark to orient himself. Like the hero of innumerable westerns, he is heading *toward* Dodge City, or like the villain of those same westerns, *away* from Dodge City, Dodge City itself serving as the solitary point on the otherwise empty and lonesome stretch of road telling the cowpoke where he is going and the villain where he has been.

What is good enough for the cowboy is good enough for the mathematician. Looking at a given line, *he* picks a point to serve as a starting spot. That point functions as an *origin,* a source of things and a center of motion. But a point, it must be remembered, is *not* a number; holding place without size and arising whimsically whenever two straight lines are crossed, it is a geometrical object, a kind of fathomless atom out of which the line is ultimately created. Analytic geometry is a program to make this desert bloom; but if arithmetic is to be found here it can only be as the result of a deliberate assignment of numbers to points, a pairing of items that are incorrigibly distinct. The mathematician thus does not discover a number at the origin: He *invokes* one. Looking out over that linear landscape, the line bisected by a point, he assigns the number 0 to the origin, if only to convey the sense on the line already conveyed in the number system itself, that at 0 things have a beginning (0, 1, 2, 3, 4,…) and at 0 they have an end (…, –4, –3, –2, –1, 0).

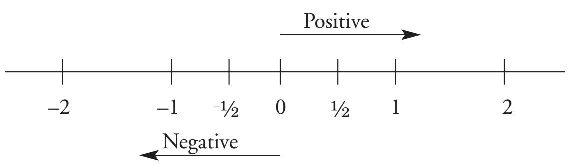
One number has been made to flower and break black blossoms on the line; the rest of them may be made to follow and crack the stony soil.

Having chosen an origin, the mathematician next chooses some fixed distance on the line to represent a unit distance. The choice of a unit is arbitrary. The distance is *fixed* because it is a measure of distance from the origin. And it is a fixed *distance* because the mathematician is measuring spatial expanse. With a unit distance thus in place, a second number makes an appearance on the line. The point precisely one unit distance from the origin is assigned the number 1. The line has now been made to blossom twice.

The number 2 blossoms on the line at the point two units from the origin, and 3 follows in turn. Every natural number is represented in just the same way. The fractions on this scheme play the role that they always play, ½, for example, denoting the point midway between 0 and 1. There are no surprises. Things are just as they seem. The scheme is simple.

If the positive integers and fractions indicate distance from the origin in one direction, the negative integers and fractions indicate distance from the origin in the other direction. It is here that the lucidity of a geometrical stage—its high desert light—may first be appreciated.

The Number Line



This elegant little exercise complete, the numbers have been inscribed on the geometric line, endowing the line with a living arithmetic content and being endowed by the line with a geometrical exoskeleton. Points on the line have now been assigned a numerical magnitude, and numbers a geometrical distance. It is possible to *measure* the distance between points and possible again to *see* the distance between numbers. Far from seeming strange, this interpretation of arithmetic and geometry strikes a deep, a resonant, chord of intuition suggesting that contrary to the historical development of these subjects, arithmetic and geometry are each aspects of a single, deeper discipline in which form and number are seamlessly matched and then merged.

The Unbearable Smoothness of Motion

And yet there is always a yet.

The geometrical line reflects the unbearable smoothness of motion perfectly; between points, there are points, those points falling in on themselves so that the line as a whole forms a *continuum,* an ancient mystic image of things at the margins of distinctness, a perfect expression of the passage we make from one place to another or from one time to another, the experience of continuity suggesting that at some level there is only *seamlessness*.

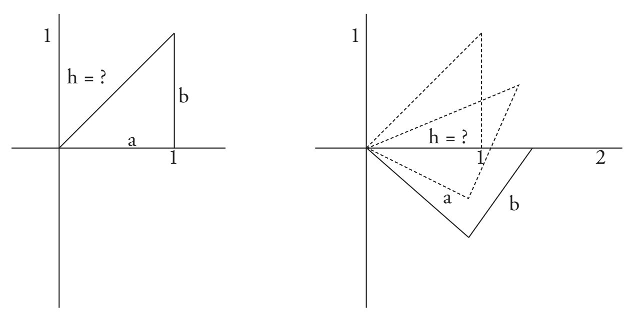
Yet the numbers are pretty hard-edged characters; each possesses a defiant sense of its own individuality, and none of them seems inclined to do much swimming toward the ocean of being. Or anything else. If points on the line find their separate identities a burden, the numbers positively *revel* in their individuality. This circumstance may provoke a squeak of suspicion, a sinister hunch that the line and the numbers inscribed upon it are in some way discordant. And although these remarks are delivered by a shrug of intuition, the shrug is backed up by an ancient argument.

The *Ifs* Accumulate

A theorem attributed to Pythagoras affirms that if **a** and **b** are the sides of a right triangle and **h** its hypotenuse, then $\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{h}^2$. The theorem embodies a striking *fact* about right triangles: whatever their particular configuration, *this* simple numerical relationship will hold among their sides. If $\mathbf{a} = 3$ and $\mathbf{b} = 4$, $\mathbf{a}^2 + \mathbf{b}^2 = 25$, and **h** must therefore be 5.

And so it is, the Pythagorean theorem embedding the waywardness of the world in an incorruptible set of conceptual constraints.

But suppose now that **a** and **b** are 1. The triangle answering to the supposition appears unremarkable. Its legs are each one unit in length. The thing seems somewhat squat. But what of **h** amid all this ordinariness? Among other things, **h** expresses the extent of a fixed and hopelessly prosaic distance in the real world. And if **h** is a distance in the real world, it is also a distance on the number line, a fact that may be seen by rotating the triangle so that its hypotenuse coincides with the axis of the number line itself. Thus inscribed on the number line, the endpoint of the hypotenuse is at precisely **h** from the origin.



So? What then *is* **h**? A distance of what *magnitude?*

It would be intellectually repugnant to learn that although **h** *is* some distance from here, it is a distance that *cannot* be correlated with any number.

To say this is to evoke one of those absurdist dramas so popular in the fifties. But it is nonetheless *appropriate* in the case of **h**, the suspicions and surmises now collecting themselves into a flat and sullen statement: *there is no way of telling.*

The overall argument is very simple, very compact, and very powerful. The Pythagorean theorem says that $\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{h}^2$, and it says so for *any* right triangle. If **a** and **b** are 1 and thus $\mathbf{a}^2 + \mathbf{b}^2 = 2$, **h** is then the number that when squared (or multiplied by itself) is 2. These trim and tidy inferential steps suffice to take the reader to the very edge of doom. If $\mathbf{a}^2 + \mathbf{b}^2 = 2$, $\mathbf{h}^2$ must be 2 and **h** itself $\sqrt{2}$.

But no such number exists.

That square root of 2 is like the Yeti or the Loch Ness monster, the snows of yesteryear, the dusky ghost of the dusty window—it is not there, it cannot be found, it is not a part of the furniture of this or any other world.

The discussion is now embedded in a tangle of concepts. Just look at this crown of thorns. *(continued)*

The square root of 2? It doesn't exist? *You're putting me on, right?*—this said with the tone of incredulity with which on ordinary occasions we treat an old friend's announcement that he is about to depart for an ashram. The thing is puzzling. It puzzled the Greeks, and it puzzled mathematicians who came after the Greeks. It puzzled mathematicians filing down the centuries, God-intoxicated Hindu sages writing in the shadows of gorgeous temples, bearded Arabic scholars fingering their caftans, profit-eyed men of the Italian renaissance.

But there it is. The ancient proof is unassailable.

Poor Fat Things

Sixteen has a square root in 4, and ¼ a square root in ½, but 2 has no square root whatsoever among the rational numbers, although it would appear that 2.25 has a square root in 1.5. *Fretfulness?* That is not quite the right word. An ancient impediment to understanding has come shambling out of the historical mists, dragging green slime behind it and snorting wetly. *Impediment?* Not quite the right word either. There are plenty of square roots beyond the square root of 2 that cannot be expressed in terms of the rational numbers—the square root of 3, for example. Like plush that under strong light reveals a series of alarming moth holes, the familiar number system is filled with strange gaps, places of reverberating emptiness. And the word for *that* is weird.

The square root of 2 forced the Greeks to the contemplation of *incommensurable magnitudes*—distances on the line that could not be correlated with any number. These are unlovely objects, those numberless distances, if only because like hairless dogs they exhibit their deficiencies so defiantly. The discovery of incommensurable magnitudes provoked a crisis among Greek mathematicians committed (as most mathematicians are) to the supremacy of numbers.

The crisis they provoked, the Greeks never resolved. In Eudoxus and in Euclid, incommensurable magnitudes make an appearance as incommensurable magnitudes, strange numberless objects. Ratios of such objects are taken and a scheme of geometry created, but in the end, there the poor fat things sit: obscure, implausible, and bizarre.

The great Hindu and Arabic mathematicians of the Middle Ages took quite another tack. Whatever incommensurable magnitudes might be, they treated such things *as if* they were really numbers—*irrational* numbers, the *irrational* a nice inadvertent touch signifying the madness loitering about the very notion—and learned many tricks by which such numbers might be manipulated. In the 12th century, for example, Bhāskara demonstrated correctly that $\sqrt{3} + \sqrt{12} = 3\sqrt{3}$. But neither Bhāskara nor anyone else ever made clear what items such as $\sqrt{3}$ *were.* The symbols resisted, as symbols so often do, any attempt to invest them with meaning. Sitting in their perfumed gardens, those thousand and one Arabian mathematicians carried out their calculations with a charming and insouciant assurance that all that gibberish actually made sense.

Not that anyone else did any better, the high medieval gibberish of Arabic mathematics appearing in Italy, France, and England as an inexpungably vital but irremediably vulgar weed. And the curious counterintuitive thing is that it didn't matter. The commonplace view of mathematics as a discipline *consecrated* to the ideal of precision has very little to do with mathematics as it is lived. Between 1500 and 1800, the great central stage of European thought is crowded with babbling and arguing figures—Cardano, Stifel, Pascal, Descartes, Wallis, Barrows, even Leibniz and the sainted Newton—saying one thing but writing quite another, agreeing in solemn convocation that irrational numbers are a fiction (almost a certain sign of bad faith, in mathematics or anything else), and then applying that fiction to numerical problems and like Bhāskara miraculously getting the answer right, the work involved in the creation of the calculus a matter evidently capable of being conducted *without* being clarified.

The straight line and the numbers themselves are somehow hopelessly discordant, the sense of dislocation all the more pressing and all the more poignant in virtue of the conviction—one shared by almost any mathematician—that the line should express the numbers and the numbers should represent the line, and that both expression and representation should be perfect and complete.

The Doctor of Discovery

In his remarkable essay, *Continuity and Irrational Numbers,* the 19th-century mathematician Richard Dedekind wrote with a sense of dawning discovery that it was severability that gave the line its essence, its past. Let us suppose, Dedekind supposed, that at a point the geometrical line is in imagination cut. The result of the cut just made is a division of the line into two segments, **A** and **B**. Every point in **A** is to the left of every point in **B**. Every point on the line determines one, and only one, cut.

It is best to think of Dedekind as a great diagnostician, a doctor of discovery. The facts are in order; but the *facts* have always been in order. The *facts* have been in plain sight for more than two thousand years. Here they are, those facts. Some distances on the line cannot be correlated with any natural or rational number. And the numbers contain gaps, places where there should be something but where there is nothing instead. There is something about the line, some kind of continuity, some special property, some thing or aspect, some feature or condition; but when it came to specifying what that thing, aspect, feature, or condition was, mathematicians were silent.

The line is in some sense richer than the numbers that are used to represent it, and this is an old, an inconvenient fact; but Dedekind's diagnosis of this problem goes beyond a revisiting of such facts in order to display the long-hidden source of the discrepancy between line and number. Every rational number, he argued, produces a cut among the numbers; but some cuts answer to no rational number and in this respect—*this alone, no other*—the numbers and the line are different. Dedekind's calm but profound investigation succeeds as an act of intellectual liberation because it connects a particular fact—that some distances cannot be measured by any rational number—with the much larger, the more general, fact that some *cuts* cannot be made at any rational number.

It is the strength of Dedekind's diagnosis that it suggests its own remedy. If the rational numbers are filled with gaps, new numbers, Dedekind urged, are needed to make good the deficiencies. Mathematicians before Dedekind had simply invoked the irrational numbers with a certain hearty carelessness, trusting in their superb intuition to get things right. In Dedekind's diagnosis, new numbers arise as the result of an *informed* act of creation.

The axiom that achieves these aims is surpassingly spare. "Whenever, then, we have to do with a cut **A** and **B**," Dedekind writes, "produced by no rational number, we *create* a new, an *irrational* number." These may seem desultory words, but Dedekind is able to paint the portrait of this new number precisely and so at least to supply the lineaments of the desired miracle. It is to be a number in set **A** greater than any other number in **A**; and thus a number less than any number in set **B**. The axiom itself serves to *compel* such a number into existence. Given any cut of the number line and, therefore, any cut of the numbers into two camps **A** and **B**, *there exists,* the axiom says—there *must* exist, the mathematician adds—one and only one number in **A** larger than any other number in **A**, the imperious *there exists* bringing something new into the world and so allowing the mathematician to share in the general mystery of creation. In the case of rational cuts, the axiom ratifies what is evident: the rational cuts are made at the numbers. But where before there was nothing more than an emptiness answering to the square root of 2, a new number now appears, a Dark Prince, an object utterly unlike any rational number, one flushed from the shadows and full of brooding mystery.

Dedekind published the results of his research in 1872 and so within the memory of the very oldest widow of a Civil War veteran, and I mention this in order to connect by some living tissue this moment with that one. The calculus had already been in existence for more than two centuries in 1872. If the calculus is much like a cathedral, its construction the work of centuries, it remained until the 19th century a cathedral suspiciously suspended in midair, the thing simply hanging there, with no one absolutely convinced that one day that gorgeous and elaborate structure would not come crashing down and fracture in a thousand pieces. Dedekind's axiom is *logically* among the fundamental affirmations of the calculus. With the axiom in place, the cathedral has a foundation. An assumption has been evoked to dispel a mystery.

Real World Rising

In the beginning, the natural numbers, 1, 2, 3, 4,....Then 0 and the negative numbers. Next, the rational numbers, or fractions. And finally the irrational numbers. I have not said what the irrational numbers *are,* only that the real number system obeys Dedekind's axiom. Like members of a goofy lodge, the other numbers express their identities unselfconsciously, but the square root of 2? It has come into existence as the result of an assumption; it stands to the other numbers in a certain relationship; when multiplied by itself it yields the number 2. But after all is said and done the thing seems determined entirely by the relationships it entertains.

A rational number or fraction, it is worthwhile to recall, enjoys a double identity, one that is on many occasions useful, as double identities often are. The number ½, for example, may be written in decimal notation as 0.5 and the number $^{15}/_{28}$ as 0.53571428571428. Now the square root of 2 may *also* be written in decimal notation, for a start as 1.414. The notation serves to restore the irrational numbers to a certain community of numbers, for, in form, 1.414 and 0.53571428571428 appear to be objects of roughly the same kind. To the extent that decimal notation serves this psychological purpose, no harm is done. But the decimal expansion of a *rational* number—the numbers after the decimal point—is either finite, as in the case of 0.5, or doomed to repeat itself after a period, and so appears among the numbers as one of those tiresome ghosts returning every Halloween to the same fireplace, where they may be found rubbing their hands and looking mournful and making clanking sounds. In the decimal expansion of $^{15}/_{28}$, the sequence 571428 occurs over and over again, clanking away.

The contrast to the irrational numbers is striking. The decimal expansion of an irrational number *never* repeats itself. Instead, the expansion trails off into the far future, each of its digits something of a surprise, the result of a unique and infinitely long object with little by way of pattern or plan to ease the understanding. The square root of 2 is 1.414, and beyond that 1.4142, and beyond that 1.414212552...; from what has gone before, there is no telling what is to come. The digits expressing this number are unpredictable, random, unique, solitary, infinite, and unfathomable. They retain an element of unavoidable mystery. Like the human soul, an irrational number is only partly known, and however more is known of either there is always infinitely more to know.

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Whatever the ultimate identity of the irrational numbers, what is known about them is of less importance than what is known of the great system in which they are embedded.

That system is *severable.* Dedekind's axiom is in force, flooding the numbers with light, flushing the irrationals from the shadows. Addition, subtraction, multiplication, and division, the immemorial operations of childhood, are entirely enabled; thus enabled, they allow the irrational numbers to function *as* numbers: $\sqrt{3} + \sqrt{12} = 3\sqrt{3}$, *because* the square root of 12 may be written as $\sqrt{4x3}$ and then as $2\sqrt{3}$, making three of those square roots in all.

The system is *ordered.* Any number if it is not equal to 0 is either greater than 0 or less than 0. It is a system in which every number finds its place and there is a place for every number.

And the system is *complete.* There are no gaps to be filled. Any cut among the numbers falls like the stroke of an ax upon a single number. Positive numbers have roots within the system. The strange black nothingness that opened up among the rational numbers is gone. Incommensurable magnitudes are no longer incommensurable. The correspondence between the geometric line and the real numbers is perfect and unblemished. □